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R. P. Boas, Jr., *Executive Editor*

Vol. 8, No. 4 **April, 1947** **pp. 189–244**

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MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
THE INSTITUTE OF MATHEMATICAL STATISTICS
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THE LONDON MATHEMATICAL SOCIETY
UNION MATEMATICA ARGENTINA

Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence 12, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence 12, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.

Mathematical Reviews

Vol. 8, No. 4 APRIL, 1947 Pages 189-244

HISTORY

Gandz, Solomon. Studies in Babylonian mathematics. III. Isoperimetric problems and the origin of the quadratic equations. Isis 32 (1940), 101–115 (1947).

[For part II cf. Isis 31, 405–425 (1940); these Rev. 7, 105.] One of the main types of Babylonian quadratic equations results from the problem of determining length x and width y if their sum a and the area xy are known. The author assumes an earlier state of development in which one experimented with numerical examples, operating with integers of the form $x=a+n$, $y=a-n$ in order to establish relations between area and perimeter. *O. Neugebauer.*

van der Waerden, B. L. Die Harmonielehre der Pythagoreer. Hermes 78, 163–199 (1943).

This is an interesting attempt to develop a consistent history of the Greek theory of harmony from Pythagoras to Ptolemy. These problems have been frequently discussed by historians of mathematics because of their importance for the development of the concepts of means, proportions and number theoretical speculations, in addition to their interest for the history of ancient physics. It is impossible to summarize the results of this paper in a short review.

O. Neugebauer (Proyidence, R. I.).

van der Waerden, B. L. Die Astronomie des Heraklides von Pontos. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. 96, 47–56 (1944).

The author gives a new interpretation of the planetary system of Heraclides Ponticus. According to this system earth, sun and planets rotate around a common center. The author's interpretation finds its confirmation in figures in several manuscripts of Chalcidius. *O. Neugebauer.*

Hofmann, Jos. E. Über Herrn Anderhubs Deutung der Theodoros-Stelle in Platons Theaetet. Deutsche Math. 7, 117–120 (1942).

The passage in question concerns the statement that Theodorus demonstrated the irrationality of $\sqrt{3}$, $\sqrt{5}$, $\cdots$, $\sqrt{17}$. A graphical procedure is suggested which would explain the stop at $\sqrt{17}$. [The author does not, however, discuss the explanation given by Hasse: H. Hasse and H. Scholz, Die Grundlagenkrisis der Griechischen Mathematik, Charlottenburg, 1928, p. 29.] It is furthermore assumed that Theodorus did not furnish a complete proof but only a plausible argument. *O. Neugebauer* (Providence, R. I.).

Reidemeister, Kurt. Das System des Aristoteles. Hamburger Math. Einzelschr. 37, 20 pp. (1943).

The author investigates the role which the Physics of Aristotle plays for his Metaphysics. He demonstrates the central position of the concepts of continuity, divisibility and movement in the system of Aristotle and discusses the difference in the attitudes of Plato and Aristotle towards the problem of mathematical reality. *O. Neugebauer.*

Maggi, A. Sui porismi d'Euclide. Period. Mat. (4) 21, 205–223 (1941).

Bell, Eric Temple. Sixes and sevens. Scripta Math. 12, 53–60 (1946).

Earlier parts of this paper appeared in Scripta Math. 9, 209–231 (1943); 10, 81–147 (1944); 11, 21–50, 139–171 (1945); these Rev. 6, 253; 7, 353. *O. Neugebauer.*

Ludendorff, Hans. Die astronomischen Inschriften in Naranjo. (Untersuchungen zur Astronomie der Maya. XIV.) Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. 1941, no. 16, 36 pp. (1942).

Dittrich, A. Die astronomischen Inschriften in Quiriguá. (Untersuchungen zur Astronomie der Maya. XV.) Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. 1942, no. 10, 51 pp. (1943).

"Aus Hans Ludendorffs Nachlass zusammengestellt."

Bjerkeseth, H. Pierre Fermat's solution of an indeterminate quadratic equation. Norsk Mat. Tidsskr. 25, 40–45 (1943). (Norwegian)

Dugas, René. L'énergie cinétique à travers l'histoire de la mécanique. Revue Sci. 84, 67–74 (1946).

Hofmann, Jos. E. Zum Winkelstreit der rheinischen Scholastiker in der ersten Hälfte des 11. Jahrhunderts. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. 1942, no. 8, 19 pp. (1942).

Hofmann, Jos. E. Über die ersten logarithmischen Rektifikationen. Eine historisch-kritische Studie in vergleichender Darstellung. Deutsche Math. 6, 283–304 (1941).

This is an interesting account of early examples of the rectification of the arc of a plane curve, between the years 1657 and 1670. It is particularly concerned with the problem of the arc of a parabola, which involves the logarithmic function, and that of the logarithmic curve $y=\log x$. The subject follows naturally after a former communication [Deutsche Math. 5, 358–375 (1940); these Rev. 7, 354] by the same author. Two points of special interest are (1) that Huygens [1657] was the first to discover the connexion between the length of a parabolic arc and the area of a hyperbola, and (2) the elucidation of the work of James Gregory [1670] rectifying the logarithmic curve, where the binomial theorem for fractional index is utilized, as also a change of variable in the integrand. The author skilfully amends an error in the source which he uses [Collins' version in Rigaud's "Correspondence of Scientific Men of the Seventeenth Century," Oxford, 1841] the original source not being accessible to him. The account confirms and in one respect expands what is published on the subject in the James Gregory tercentenary memorial volume of the Royal Society of Edinburgh [Bell, London, 1937; these Rev. 1, 129]. Full justice is done to Gregory as an independent discoverer of the binomial theorem. *H. W. Turnbull.*

Hofmann, Jos. E. Das Opus Geometricum des Gregorius a S. Vincentio und seine Einwirkung auf Leibniz. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1941**, no. 13, 80 pp. (1 plate) (1942).

Hofmann, Jos. E. Studien zur Vorgeschichte des Prioritätstreites zwischen Leibniz und Newton um die Entdeckung der höheren Analysis. I. Materialien zur ersten mathematischen Schaffensperiode Newtons (1665–1675). Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1943**, no. 2, 130 pp. (2 plates) (1943).

Fleckenstein, J. O. Johann I Bernoulli als Kritiker der "Principia" Newtons. Elemente der Math. **1**, 100–108 (1946).

Ortvay, R. Isaak Newton und die Wissenschaft der Gegenwart. Anlässlich der 300. Jahreswende seiner Geburt. Mat. Fiz. Lapok **50**, 262–289 (1 plate) (1943). (Hungarian. German summary)

Ortvay, R. Galilei und die Entfaltung des neuzeitlichen wissenschaftlichen Denkens. (Zur 300. Jahreswende des Todes von Galileo Galilei). Mat. Fiz. Lapok **49**, 139–169 (1 plate) (1942). (Hungarian. German summary)

Masotti, A. Maria Gaetana Agnesi. Rend. Sem. Mat. Fis. Milano **14**, 89–127 (4 plates) (1940).

Heffter, Lothar. Obituary: Oskar Bolza. Jber. Deutsch. Math. Verein. **53**, 2–13 (1943).

Bompiani, E. Obituary: Enea Bortolotti. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 231–240 (1942).

Löffler, Eugen. Alexander von Brill. Zur 100. Wiederkehr seines Geburtstages. Jber. Deutsch. Math. Verein. **53**, 82–89 (1943).

Tonolo, Angelo. Obituary: Annibale Comessatti. Rend. Sem. Mat. Univ. Padova **15**, 1–7 (1946).

Heegaard, Poul. Obituary: Friedrich Engel. Norsk Mat. Tidsskr. **23**, 129–131 (1941). (Norwegian)

Nagell, T. Obituary: Erik Holmgren. Norsk Mat. Tidsskr. **26**, 1–2 (1944). (Norwegian)

Zigány, Ferenc. The work of Lipót Klug. Mat. Fiz. Lapok **50**, 205–222 (1943). (Hungarian)

Johansson, Ingebrigt. Sophus Lie. Remarks on the occasion of the centenary of his birth. Norsk Mat. Tidsskr. **24**, 97–106 (1 plate) (1942). (Norwegian)

Picone, Mauro. Obituary: Ernesto Pascal. Rend. Accad. Sci. Fis. Mat. Napoli (4) **12**, 54–82 (1 plate) (1942).

Strubecker, Karl. Obituary: Ernst August Weiss. Deutsche Math. **7**, 254–298 (1943).

ALGEBRA

Waldapfel, L. Über das Profil der Permutationen. Mat. Fiz. Lapok **50**, 257–261 (1943). (Hungarian. German summary)

To every permutation of $1, 2, \cdots, n+1$, we attach a corresponding sequence of n + and − signs as follows. Let $a_1, \cdots, a_{n+1}$ be any permutation; the ith symbol is + if $a_i < a_{i+1}$ and − otherwise. The author investigates the number of permutations to which the same sequence corresponds and obtains a recursion formula. *P. Erdős.*

Amici, Andrea. Sul determinante di Stern ed i coefficienti polinomiali. Atti Accad. Ligure **2**, 183–186 (1942).

Upon equating the results obtained by developing the Stern determinant $|C_{i-1}^{j-1}|$, $i, j = 1, \cdots, n$ (C_{i-1}^{j-1} has its usual combinatory significance), in two different ways, the author derives the formula

$$\sum(-1)^i(k; x_{i_1}, x_{i_2}-1, \cdots, x_{i_n}-n+1) = k!V/x_1!x_2!\cdots x_n!,$$

where $i_1, i_2, \cdots, i_n$ is a permutation of $1, 2, \cdots, n$ with i inversions, $x_{i_1}+x_{i_2}-1+\cdots+x_{i_n}-n+1=k$, V is the Vandermonde determinant of $x_1, x_2, \cdots, x_n$ and the parenthesis in the summation denotes the polynomial coefficient of base k and indices as exhibited. *L. M. Blumenthal.*

Finney, D. J. Latin squares of the sixth order. Experientia **2**, 404–405 (1946).

This is a preliminary communication giving enumerations of certain partitions of 6×6 Latin squares. A partition is a set of distinct parts of the square, part k having k letters in each row and column and each letter k times. The partitions disclose symmetry and other properties of a square and may be used as makeshifts for the missing orthogonal (Graeco-Latin) squares: partition (1^6). *J. Riordan.*

Gavrilov, M. A. Determination of the number of contacts in relay-contact decoders, and their distribution among the relays. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1945**, 1109–1127 (1945). (Russian)

The problem of the paper has to do with the design of switching circuits for some sort of a relay decoding device not specified or described. Although the author speaks of relays, the problem can be stated in terms of multiple switches, either single or double throw. In the logically simplest case of the "pyramidal" switching layout one has n relays; the first has two contacts, the second has four contacts, the nth has 2^n. There are 2^n ways in which one may decide whether to energize or not each of the relays and these correspond to 2^n outputs from the nth relay. With this scheme one minimizes the number of relays employed to give a certain number of outputs. However, this puts an unfair burden on the nth relay, which may have to have an uneconomically large number of contacts. The author's schemes call for a more uniform distribution of the contacts among the several relays, and the consequent use of more relays with fewer contacts per relay. The method for devising these schemes is nontentative and is a type of Boolean algebra. Enumeration formulas are devised for the number of contacts. The presentation would have been much clearer had the author included a careful description of the types of relays considered. *D. H. Lehmer.*

Løvestad, Ludvig. A property of scales of measurement. Norsk Mat. Tidsskr. **26**, 9–13 (1944). (Norwegian)

Consider a set with elements x_k, $k = 0, \pm1, \pm2, \cdots$ and suppose that for every pair (i, k) there is defined a number

x_{ik}. Suppose furthermore that there exists a function f, such that $x_{pr}=f(x_{pk}, x_{rk}, a_{rk})$ for all combinations of subscripts, where a_{rk} depends only on r and k. Some conditions on f are stated and special cases considered. *W. Feller.*

Conte, Luigi. Le formule di Girard-Newton. Period. Mat. (4) **21**, 224–233 (1941).

Ziaud-Din, M. Tables of symmetric functions for statistical purposes. Proc. Nat. Acad. Sci. India. Sect. A. **10**, 53–60 (1940).

Stressing their value for statistical purposes, the author tabulates for $n=7, 8, 9$ the symmetric partition functions $P(\mu)$, denoted by $(\mu)=(1^\alpha 2^\beta 3^\gamma \cdots)$, as linear combinations of products $S(\mu)=S_1^\alpha S_2^\beta S_3^\gamma \cdots$ of the power sums S_r. The coefficients are closely related to the characters of the symmetric group, published for $n=1$ to 9 by Littlewood and Richardson [Philos. Trans. Roy. Soc. London. Ser. A. **233**, 99–141 (1934)], for $n=10$ by Littlewood [Proc. London Math. Soc. (2) **39**, 150–199 (1935)] and for $n=11, 12, 13$ by the author [Ann. Math. Statistics **9**, 63–65 (1938); Proc. London Math. Soc. (2) **39**, 200–204 (1935); **42**, 340–355 (1937)]. It may be noted that the coefficient $C(\mu, \nu)$ in the formula $P(\mu)=\sum_\nu C(\mu, \nu)S(\nu)$ could be obtained as follows: $C(\mu, \nu)=\sum_\lambda \phi(\lambda)F(\mu, \lambda, \nu)/D(\mu)$, where μ, λ, ν are partitions and $\phi(1^\alpha 2^\beta 3^\gamma 4^\delta \cdots)=(0!)^\alpha(-1!)^\beta(2!)^\gamma(-3!)^\delta \cdots$, $D=\alpha!\beta!\gamma!\delta! \cdots$, and $F(\mu, \lambda, \nu)$ is the number of ways the given partition μ of n into m positive integral summands (written on m slips of paper) can be sorted into envelopes according to the partition λ of m, so that the sums within the envelopes give the partition ν of n. The table for degree 9 contains numerous errors, including at least eight for $\mu=(4\cdot 1^5)$. Taking $\mu=\nu$ and $\lambda=1^6$ we should have $C(\mu, \mu)=(0!)^6 1/5!=1/120$, instead of 0. Also for $\mu=(3^2 2\cdot 1)$ and $\nu=1\cdot 8$, we should have $\lambda=1\cdot 3$, $\phi(\lambda)=2$, $D(\mu)=2$, $C(\mu, \nu)=2/2$, not 4/2 as printed. *J. S. Frame.*

Kerawala, S. M. Table of monomial symmetric functions of weight 9. Proc. Nat. Acad. Sci. India. Sect. A. **11**, 51–55 (1941).

The author corrects 26 errors in the table of Ziaud-Din reviewed above, and arranges the partitions in decreasing dictionary order on the left column from top to bottom (for μ) and on the top row from right to left (for ν). The table is printed clearly and is easy to use. *J. S. Frame.*

Kerawala, S. M., and Hanafi, A. R. The table of symmetric functions of weight 10. Proc. Nat. Acad. Sci. India. Sect. A. **11**, 56–63 (1941).

The table of symmetric functions of weight 10 is constructed from already published tables of lower weights [see the preceding review]. The digits in the coefficients of $2\cdot 3\cdot 5$, $4\cdot 6$, $3\cdot 7$ in the expansion of (1^{10}) should all be preceded by 1's, and the coefficients of $1^2\cdot 2^4$ and $1^2\cdot 4^2$ in the expansion of $(2^2\cdot 1^6)$ should be 45 and 450 instead of 135 and 360. *J. S. Frame* (East Lansing, Mich.).

Kerawala, S. M., and Hanafi, A. R. Table of monomial symmetric functions of weight 11. Proc. Nat. Acad. Sci. India. Sect. A. **12**, 81–96 (1942).

The table of symmetric functions of weight 11 is constructed and errors are noted in the table of weight 10 as mentioned in the preceding review. *J. S. Frame.*

Coble, Arthur B. Ternary and quaternary elimination. Amer. J. Math. **68**, 521–543 (1946).

The resultant R of n forms in n variables, $(\alpha_i)^{l_i}$, had been obtained previously by Macaulay and van der Waerden as the G.C.D. of $(l+1)$ determinants found in the "dialytic array" of forms of order $(l+1)$, where $l=\sum(l_i-1)$. The result is precise, but neither the determinants nor the extraneous factor in each is invariantive and the determinants are of high order. Extending a method initiated by F. Morley [same J. **47**, 91–97 (1925)] and developed by Morley and Coble [same J. **49**, 463–488 (1927)] employing an identity obtained from the coordinates of a common point, the author obtains considerably simpler determinants which are invariantive, for many ternary and quaternary cases.

Thus in the ternary case, when the degrees are 2, 3, 7, a determinant of order 30 would result, which is a multiple of the resultant R by a given extraneous factor. This can be compared with van der Waerden's R as the G.C.D. of three determinants of order 66, or Macaulay's R as the quotient of two determinants of respective orders 66 and 25.

There is some discussion of the unique form of class l which is apolar to each of the given forms. The method is applicable in higher domains but its flexibility rapidly diminishes. *D. E. Littlewood* (Swansea).

Coble, Arthur B. On the expression of an algebraic form in terms of a set of forms with non-zero resultant. Amer. J. Math. **68**, 544–552 (1946).

If the resultant R of a set of n forms $f_i=(\alpha_\mathcal{Y})^{l_i}$ in n variables is not zero, it is well known that an algebraic form of order $m>l=\sum(l_i-1)$ is a member of the module defined by the given forms, i.e., that $R(\delta y)^m=\sum_{i=1}^n(\beta_\mathcal{Y})^{m-l_i}(\alpha_\mathcal{Y})^{l_i}$. Ordinarily, no attention is paid to the coefficients $(\beta_\mathcal{Y})^{m-l_i}$. This article obtains explicit values for these coefficients. An extension of the module is also obtained which yields similar expressions for $R(\delta y)^m$ when $0<m\leqq l$, in which the resultant R on the left is balanced on the right by the coefficients of the unique form $A=(a\eta)^l$ which is apolar to each of the given forms. Much of the paper is devoted to a detailed account of this apolar form, which can define the module. It is proved that the apolarity invariant of this apolar form and of the Jacobian of the n forms is the resultant R of the n forms. *D. E. Littlewood* (Swansea).

Wendelin, H. Ein Determinantenentwicklungssatz und seine Anwendung in der Theorie der Integralgleichungen. Deutsche Math. **6**, 267–271 (1941).

Let D denote the determinant $|x_{\mu\nu}|$, $\mu, \nu=0, 1, \cdots, n$; let s be a fixed integer in the set $0, 1, \cdots, n$; and let $\Delta(s\sigma_1 \cdots \sigma_{r-1})$ denote the minor determinant of D obtained by striking out the rows and columns with indices $s, \sigma_1, \cdots, \sigma_{r-1}$. The author establishes the expansion theorem

$$D=\sum_{r=1}^{n+1}(-1)^{r-1}\sum_{\{\sigma_1 \cdots \sigma_{r-1}\}}{}'\,\Delta(s\sigma_1 \cdots \sigma_{r-1})\sum_{(\rho_1 \cdots \rho_{r-1})\sigma}{}''\,x_{s\rho_1}x_{\rho_1\rho_2}\cdots x_{\rho_{r-1}\sigma},$$

where $\sum'$ denotes the sum over all combinations $\sigma_1, \cdots, \sigma_{r-1}$ selected from $\{0, 1, \cdots, n\}-\{s\}$ and $\sum''$ denotes the sum over all permutations $(\rho_1 \cdots \rho_{r-1})$ of $\sigma_1, \cdots, \sigma_{r-1}$. As an application of this expansion, the author gives an elementary proof of the known theorem that the Fredholm and Neumann forms of the resolvant kernel of a nonhomogeneous linear integral equation are formally identical.

G. B. Price (Lawrence, Kan.).

Giuga, Giuseppe. Alcune estensioni del determinante di Vandermonde. Rend. Accad. Sci. Fis. Mat. Napoli (4) **12**, 263–296 (1942).

This is a collection and systematic development of many known results on the determinants of Vandermonde, Stern and Trudi and their generalizations, mostly determinants whose elements are binomial coefficients involving integral parameters, together with extensions by the author. In all, 34 types of determinant are considered.

C. C. MacDuffee (Río Piedras, P. R.).

Neiss, F. Eine Bemerkung zur Hauptachsentransformation. Deutsche Math. **7**, 81–83 (1942).

The author outlines an elementary and self-contained process for reducing a real symmetric matrix to diagonal form by means of orthogonal transformations.

C. C. MacDuffee (Río Piedras, P. R.).

Lee, H. C., and Liang, S. L. Some properties of Hermitian matrices. Acad. Sinica Science Record **1**, 321–324 (1945).

Let A be a Hermitian matrix whose principal minor of order r in the upper left corner has the determinant Δ_r, and suppose each $\Delta_r \neq 0$. An explicit formula is obtained for a triangular matrix T such that $A = TDT^*$, where D is diagonal with its jth diagonal element $+1$ or -1 according as $\Delta_{j-1}\Delta_j$ is positive or negative. This formula yields an identity of Lagrange when A is positive definite.

C. C. MacDuffee (Río Piedras, P. R.).

Palermo, Francesco. Sull'equazione minima di una matrice. Boll. Accad. Gioenia Sci. Nat. Catania (3) **18** (1941), 26–32 (1942).

The author proves the well-known formula for the minimal equation of a matrix in terms of its elementary divisors. [Cf. Wedderburn, Lectures on Matrices, Amer. Math. Soc. Colloquium Publ., v. 17, New York, 1934, p. 42.]

D. E. Rutherford (St. Andrews).

Brauer, Alfred. Limits for the characteristic roots of a matrix. Duke Math. J. **13**, 387–395 (1946).

Let $A = (a_{ij})$ be a square matrix of order n over the complex field with characteristic roots ω_i and let $R_i = \sum_{j=1}^{n} |a_{ij}|$, $S_j = \sum_{i=1}^{n} |a_{ij}|$, $P_i = R_i - |a_{ii}|$, $Q_j = S_j - |a_{jj}|$. If $(x_1, \cdots, x_n)$ is a characteristic vector of A belonging to the characteristic root ω and if $|x_r| \geqq |x_i|$, $i = 1, \cdots, n$, $|\omega - a_{rr}|\,|x_r| \leqq \sum_{j=1}'^{n} |a_{rj}|\,|x_j| \leqq |x_r| P_r$, where $\sum'$ omits $j = r$, and accordingly $|\omega - a_{rr}| \leqq P_r$. Thus the author proves the theorem that each characteristic root of A lies in at least one of the circles $|z - a_{ii}| \leqq P_i$ and in at least one of the circles $|z - a_{jj}| \leqq Q_j$. If R is the maximum of the R_i and S the maximum of the S_j, it follows that $|\omega_i| \leqq \min(R, T)$ and, if $d = \min(|a_{ii}| - P_i) > 0$, $i = 1, \cdots, n$, that $|\omega_i| \geqq d$ and that the absolute value of the determinant of A is not less than $d^n > 0$. The author then shows by similar reasoning that $|\omega_i| = R$ for one value of i, if and only if $R_1 = \cdots = R_n$ and $\arg a_{ij} = \phi + \phi_i - \phi_j$, where ϕ and $\phi_1, \phi_2, \cdots, \phi_n$ are arbitrary real numbers. He also proves the theorem: if the main diagonal elements and the coefficients of the characteristic equation of A are real and if, for each i and j, $|a_{ii} - a_{jj}| > P_i + P_j$, all the characteristic roots of A are real and distinct.

J. Williamson (Flushing, N. Y.).

Vogel, Alfred. Das charakteristische Polynom einer Matrix und die Trennung seiner Wurzeln. Deutsche Math. **7**, 521–552 (1944).

The author gives a method for computing constants which separate the roots of the characteristic equation of an $n \times n$ matrix for $n = 2, 3, 4$. These constants are expressed in terms of the elements of a somewhat complicated form to which the original matrix can be reduced by similarity transformations.

C. C. MacDuffee (Río Piedras, P. R.).

Mignosi, Giuseppe. Sulle caratteristiche delle matrici. Rend. Sem. Mat. Univ. Padova **13**, 26–29 (1942).

A determination of conditions which imply that the real part of each characteristic root of a complex matrix is positive (negative).

N. H. McCoy (Northampton, Mass.).

Cherubino, Salvatore. Sopra un certo pfaffiano. Rend. Sem. Mat. Univ. Padova **13**, 30–35 (1942).

A determination of the Pfaffian of the matrix $A - \rho I_0$, where A is a skew-symmetric matrix of order $2p$, ρ is an indeterminate and

$$I_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

I being the unit matrix of order p.

N. H. McCoy.

Amato, V. Sulle matrici invarianti proprie delle sottoalgebre fondamentali dell'algebra delle matrici complesse di ordine n. Atti Accad. Gioenia Catania (6) **4**, no. XII, 7 pp. (1940).

Amato, V. Sui polinomi di matrice. Atti Accad. Gioenia Catania (6) **5**, no. IV, 5 pp. (1942).

Let A_x be the algebra of all complex matrices of order n which commute with the given matrix x, and $p(x)$ a scalar polynomial in x. In both papers the author determines necessary and sufficient conditions that $A_{p(x)}$ coincide with A_x.

N. H. McCoy (Northampton, Mass.).

Abstract Algebra

★**Dubreil, Paul. Algèbre. Tome I. Équivalences, Opérations, Groupes, Anneaux, Corps.** Gauthier-Villars, Paris, 1946. x+305 pp.

This lucid and simple introduction to abstract algebra approaches the subject from an extremely general point of view. Thus it begins with a general discussion of sets and relations. The concepts of equivalence relations and of "associable" or permutable equivalence relations [cf. P. Dubreil and M.-L. Dubreil-Jacotin, J. Math. Pures Appl. (9) **18**, 63–95 (1939), in particular, p. 75] are also treated in general. Various postulate systems for groups and their generalizations (groupoids, demigroups, etc.) are compared at length. Right- and left-substitution and cancellation laws are discussed [cf. P. Dubreil, Mem. Acad. Sci. Paris **63**, no. 3 (1941); these Rev. **8**, 15 for further results]. Ordered fields and domains are also discussed very carefully, with especial reference to completeness properties. The book concludes with an exposition of unique factorization, Euclidean domains, polynomial rings, perfect fields, symmetric polynomials and related topics. There are aiso appendices on cardinal numbers, the axiom of choice, abstract spaces and one-to-one correspondences.

G. Birkhoff.

Birkhoff, George D., and Birkhoff, Garrett. Distributive postulates for systems like Boolean algebras. Trans. Amer. Math. Soc. **60**, 3–11 (1946).

M. H. A. Newman's characterization of Boolean algebras and rings, without associative assumptions [J. London Math. Soc. **16**, 256–272 (1941); these Rev. **4**, 70] is simplified and extended to a more general class of (not

necessarily associative) algebras. The authors study complemented distributed systems (c.d.s.), that is, systems with double composition satisfying (P_1) $a(b+c)=ab+ac$; (P_1') $(a+b)c=ac+bc$; (P_2) there exists 1, with $a1=a$; (P_3) there exists 0, with $a+0=a$; (P_3') $0+a=a$; (P_4) to each a there exists at least one a' with $aa'=0$ and $a+a'=1$.

In a c.d.s. it is proved that: a' is unique and involutory, $a''=a$; $a^2=a$; $a0=0a=0$; complete left right symmetry in the properties of $+$ and of $\times$ exists, e.g., $1a=a$, $a+0=a$. The following Newman structure theorem is proved. In a c.d.s. each element z may be uniquely expressed as $z=x+y$, the sum of an "even" x (additively idempotent element, $x+x=x$), and an "odd" y (additively nilpotent, $y+y=0$); this representation is direct, that is,

$$\left.\begin{matrix} z_1=x_1+y_1, \\ z_2=x_2+y_2 \end{matrix}\right\} \rightarrow \begin{cases} z_1+z_2=(x_1+x_2)+(y_1+y_2), \\ z_1z_2=x_1x_2+y_1y_2. \end{cases}$$

It is shown that the assumptions (T) $a^2=a$, (T') $a+1=1+a$ (provable in any c.d.s. provided a is even), together with (P_1)-(P_1'), (P_2)-(P_2') characterize a distributive lattice with 1; a corollary of this yields a new proof of Newman's result that the subset of all even elements forms a Boolean algebra.

The last part of corollary 1, § 2, should evidently read: $a'+a=1$. *A. L. Foster* (Berkeley, Calif.).

Bischof, Artur. Beiträge zur Carathéodoryschen Algebraisierung des Integralbegriffs. Schr. Math. Inst. u. Inst. Angew. Math. Univ. Berlin **5**, 237–262 (1941).

Let $\mathfrak{L}$ be a partially ordered set with a least element 0; two elements of $\mathfrak{L}$ are called disjoint if 0 is the only element smaller than both of them. Consider the condition: (*) for any two distinct elements of $\mathfrak{L}$ there exists a nonzero element smaller than one of them and disjoint from the other. The author proves (assuming (*)) that (T): $\mathfrak{L}$ may be embedded in a complete Boolean ring $\mathfrak{R}(\mathfrak{L})$ which is such that if $\mathfrak{L}$ is a multiplicative system (i.e., if there exists in $\mathfrak{L}$ a commutative and idempotent multiplication in terms of which the partial ordering of $\mathfrak{L}$ is defined in the usual manner) then $\mathfrak{R}(\mathfrak{L})$ is the smallest complete Boolean ring containing $\mathfrak{L}$, and if $\mathfrak{L}$ is homomorphic to $\mathfrak{L}'$ then $\mathfrak{R}(\mathfrak{L})$ is homomorphic to $\mathfrak{R}(\mathfrak{L}')$. If $\mathfrak{L}$ happens to be a Boolean ring then as a multiplicative system $\mathfrak{L}$ is clearly isomorphic to a subsystem of the class of all subsets of $\mathfrak{L}$ ($A_0 \varepsilon \mathfrak{L}$ corresponds to the set $\{A : A \subseteq A_0\}$); from this remark and (T), Stone's theorem on the representation of Boolean rings follows immediately.

By making use of Stone's theorem the author gives a proof of the following statement. If $\mathfrak{R}$ is a complete Boolean ring and m is a positive, countably additive, finite measure on $\mathfrak{R}$, then there exists a Boolean ring $\mathfrak{R}'$ of subsets of some set, and a nonnegative, countably additive finite measure m' on $\mathfrak{R}'$ such that the reduced measure algebra $\{\mathfrak{R}'/\mathfrak{N}', m'\}$, where $\mathfrak{N}'$ is the class of all sets of m' measure zero, is isomorphic to $\{\mathfrak{R}, m\}$.

A point function f (Ortsfunktion) with domain D in a complete Boolean ring $\mathfrak{R}$, $D \varepsilon \mathfrak{R}$, is defined as a pair (α, β) of real functions defined for each $A \subseteq D$ ($A \varepsilon \mathfrak{R}$, $A \neq 0$) satisfying the conditions $\alpha(A)=\inf\{\beta(B) : B \subseteq A\}$, $\beta(A)=\sup\{\alpha(B) : B \subseteq A\}$. This definition is a modification of one given by Carathéodory [Math. Z. **46**, 181–189 (1940); these Rev. **1**, 304]; $\alpha(A)$ and $\beta(A)$ are to be thought of as the lower and upper bounds, respectively, of the function f on the "set" A. In terms of this definition the author proves the analogue of the Radon-Nikodym theorem on the representation of absolutely continuous set functions by integrals.

The paper concludes with a definition of direct product for measure algebras. *P. R. Halmos* (Chicago, Ill.).

Tarski, Alfred. Ideale in vollständigen Mengenkörpern. II. Fund. Math. **33**, 51–65 (1945).

The first part appeared in Fund. Math. **32**, 45–63 (1939). If p is any cardinal number, Tarski calls an ideal J of a Boolean algebra L p-saturated if and only if $L-J$ contains no subset S of p elements such that $x \cap y \varepsilon J$ for all pairs $x, y \varepsilon S$. The 2-saturated ideals are the prime ideals and L itself. He shows that there are no $\aleph_0$-saturated ideals which are not already n-saturated for some finite n. He notes that the "atoms" of L generate an ideal P. For L to possess a finitely additive measure function, zero on all atoms, and assuming only finitely many values, it is necessary and sufficient that P is contained in a $\aleph_0$-saturated ideal. He shows that L possesses a finitely additive measure function, zero on all atoms, if and only if it contains a countable set of disjoint elements, and that in this case the measure function can be made to assume only the values 0 and 1. He concludes with some topological applications.

G. Birkhoff (Cambridge, Mass.).

Pospíšil, Bedřich. Wesentliche Primideale in vollständigen Ringen. Fund. Math. **33**, 66–74 (1945).

The author considers the ideals of the Boolean algebra $L(\alpha)$ of all subsets of a fixed class of infinite cardinal number α. He shows that $L(\alpha)$ contains 2^{2^α} prime ideals whose complements contain α points. One can even select 2^{2^α} such prime ideals, such that (i) every selected ideal has 2^α generators and (ii) any two selected ideals generate $L(\alpha)$.

G. Birkhoff (Cambridge, Mass.).

Levitzki, Yaakov. On powers with transfinite exponents. Riveon Lematematika **1**, 8–13 (1946). (Hebrew)

Let S be a system with an associative multiplication containing an element O such that $Os=sO=O$ for every $s \varepsilon S$. An ideal in S is a subset A such that $as \varepsilon A$, $sa \varepsilon A$ for $a \varepsilon A$, $s \varepsilon S$. For some ordinal number ρ, $A^\rho=A^\tau$ for all $\tau > \rho$, and A^ρ is called the "first kernel" of A; an example is given to show that this need not be idempotent. Iteration of first kernel formation leads to a transfinite sequence of kernels which is ultimately constant and whose limit is the "ultimate kernel" of A; this is idempotent. The ultimate kernel of S is the logical sum of all the idempotent ideals, and a suitably defined residual system of S relative to this kernel has no idempotent ideals. Applications of these methods will be given elsewhere. *I. S. Cohen* (Philadelphia, Pa.).

Andreoli, Giulio. Equazioni algebriche nei corpi numerici. Rend. Accad. Sci. Fis. Mat. Napoli (4) **12**, 234–244 (1942).

An exposition of elementary properties of algebraic adjunctions to finite fields. *N. H. McCoy.*

Schafer, R. D. Equivalence in a class of division algebras of order 16. Bull. Amer. Math. Soc. **52**, 874–881 (1946).

Let L be a linear space of order eight over F and define two Cayley-Dickson division algebras C and C_0, whose elements are the vectors of L, and whose products are designated, respectively, by ab and $a \cdot b$. Each of the algebras then has an involution $a \rightarrow \bar{a}$ and we may construct an algebra A of order sixteen whose elements are pairs (a, b) of vectors of L with multiplication defined by $(a, b)(x, y)=(ax+y \cdot \bar{b}, \bar{a}y+xb)$. The author gives conditions for A to be a division algebra and shows that a second algebra A^*, based on a space L^* and algebras C^*, C_0^*, is isomorphic to A if and only if there are a one-to-one mapping $x \rightarrow xP$ of L^* on L and a nonzero element δ of F such that P is an isomorphism of C^* and C, and $\delta^2 P$ is an iso-

morphism of C_0^* and C_0. The resulting conditions for a non-singular linear transformation of A to be an automorphism of A are then determined. *A. A. Albert* (Chicago, Ill.).

Madhavarao, B. S., Thiruvenkatachar, V. R., and Venkatachaliengar, K. Algebra related to elementary particles of spin $\frac{3}{2}$. Proc. Roy. Soc. London. Ser. A. **187**, 385–397 (1946).

The authors investigate the algebra generated by the unit element 1 together with the four matrices β_μ which occur in the relativistic wave equation of a particle of half-odd integral spin, on the basis of the general commutation rules for such matrices given by one of the authors [Madhavarao, Proc. Indian Acad. Sci., Sect. A. **15**, 139–147 (1942); these Rev. **4**, 31]. They first show that this algebra is in general a direct product of a Dirac algebra (of rank 16) and an algebra which they call a ξ-algebra. For the special case of spin $\frac{3}{2}$ they show that this ξ-algebra is of rank 42, and that it has just three inequivalent irreducible matrix representations, of degrees 1, 4, and 5 (note that $1^2+4^2+5^2=42$). These representations are worked out explicitly, and a list of traces is also given. The algebra generated by the β_μ is of rank $672=16\times 42$. These results may prove useful in connection with Bhabha's theory of the proton as a particle with possible maximal spin $\frac{3}{2}$ [Proc. Indian Acad. Sci., Sect. A. **21**, 241–264 (1945); these Rev. **7**, 538]. *O. Frink.*

NUMBER THEORY

Copeland, Arthur H., and Erdös, Paul. Note on normal numbers. Bull. Amer. Math. Soc. **52**, 857–860 (1946).

D. G. Champernowne proved [J. London Math. Soc. **8**, 254–260 (1933)] that the infinite decimal $0.123456789101112\cdots$ is normal in the scale of ten. He conjectured that the infinite decimal $0.12357111317\cdots$, formed by the sequence of primes, has the same property. This note contains a proof of the following more general result, which includes the conjecture of Champernowne as a special case. If $a_1, a_2, \cdots$ is an increasing sequence of integers such that, for every $\theta<1$, the number of a's up to N exceeds N^θ provided N is sufficiently large, then the infinite decimal $0.a_1a_2a_3\cdots$ is normal in the scale in which these integers are expressed. The proof of this result is based on the concept of (ϵ, k) normality due to Besicovitch [Math. Z. **39**, 146–156 (1934)]. *R. D. James* (Vancouver, B. C.).

Pisot, Charles. Répartition (mod 1) des puissances successives des nombres réels. Comment. Math. Helv. **19**, 153–160 (1946).

Let $\alpha>1$ and λ be real, let $(\lambda\alpha^n)$ be the fractional part of $\lambda\alpha^n$ $(n=0, 1, 2, \cdots)$ and let E be the set of limit points of $(\lambda\alpha^n)$. The paper consists partly in the exposition of the following known results, some of them due to the author [Ann. Scuola Norm. Super. Pisa (2) **7**, 205–248 (1938)], some due to Vijayaraghavan [Proc. Cambridge Philos. Soc. **37**, 349–357 (1941); these Rev. **3**, 274]. (I) The set of all numbers α and λ such that E consists of a finite number of points (briefly: E is finite) is denumerable. (II) If α is algebraic, necessary and sufficient conditions for E to be finite are that α is an algebraic integer whose conjugates have their moduli all less than 1, and that λ is an algebraic number of the field of α. In addition the following theorem is proved. (III) If E is finite and k denotes the number of irrational numbers in E and if the convergence of $(\lambda\alpha^n)$ towards its limit points is $o(n^{-k-1})$, then α is algebraic and therefore α and λ satisfy the conditions of (II). The proof of (III) is based on a method already used by the author in a previous paper on criteria for algebraic numbers [see these Rev. **4**, 266]. *R. Salem* (Cambridge, Mass.).

Ostmann, Hans-Heinrich. Beweis einer Vermutung über die asymptotische Dichte und Verschärfung einer Abschätzung für die Dichte der Summe zweier Zahlenmengen. Deutsche Math. **6**, 213–247 (1941).

Let $A, B, \cdots$ be sets of nonnegative integers not necessarily containing the number 0, $A(n)$ the number of positive integers not exceeding n in A. The author introduces the quantities

$$\sigma=\inf_{n\geq 0}\frac{A(n)+B(n)}{n},\quad \sigma_*=\inf_{n\geq 0}\frac{A(n)+B(n)+1}{n+1},\quad \frac{A(0)+B(0)}{0}=1.$$

Let j be the smallest integer for which $A(n)+B(n)-n=-1$. The author gives estimates of the density γ of $C=A+B$ and of the quantity $\inf (C(n)+1)/(n+1)$ in terms of σ, σ_* and j; in some cases these are sharper than the results known at the time of publication of his paper.

Let

$$\gamma^*=\liminf C(n)/n,\quad \bar{\gamma}^*=\limsup C(n)/n,$$
$$\sigma^*=\min\{\liminf (A(n)+B(n))/n, 1\}.$$

The author proves that $\gamma^*\geqq(j/(j+1))\sigma^*$ and also proves that this estimate is in a sense sharp. Let j_n be the smallest nonnegative integer for which $A(n)+B(n)-n=-(u-1)$ and let $\tau^*=\limsup\{A(j_k)+B(j_k)\}/j^k$ if there are infinitely many j_k, $\tau^*=\sigma^*$ otherwise. Then either $\bar{\gamma}^*\geqq(j/(j+1))\tau^*$ or $\gamma^*=1$. *H. B. Mann* (Columbus, Ohio).

Tambs Lyche, R. A property of the symmetric functions of the first n odd numbers. Norsk Mat. Tidsskr. **26**, 96–103 (1944). (Norwegian)

Cf. Avh. Norske Vid. Akad. Oslo. I. **1944**, no. 9 (1945); these Rev. **7**, 505.

Zahlen, Jean Pierre. Sur un genre nouveau de critères de primalité. Euclides, Madrid **6**, no. 64, 380–387 (1946).

In an earlier paper the author has generalized Wilson's theorem as follows. The integer p is a prime if and only if (1) $k!(p-k-1)!\equiv(-1)^{k+1}\pmod p$ for $0\leqq k\leqq p-1$. He now uses (1) to obtain a criterion for the fact that p and $p+d$ are both primes. [The criterion (1) was proved by Genty in 1783 and the well-known special case $k=(p-1)/2$ was given by Lagrange in 1771 [cf. Dickson, History of the Theory of Numbers, v. 1, Washington, 1919, pp. 64, 62].] The author's criterion is an immediate corollary of (1) and is as impracticable as (1) for numerical calculations. The author's result for $d=2$ is obtained more simply from

$$(2)\quad \frac{(p-1)!+1}{p}\equiv\frac{(p+1)!+(-1)(-2)}{(-2)(-1)(-2)}\equiv\frac{-1+2}{-4}\equiv-\frac{1}{4}\pmod{p+2}.$$

If p is a prime, then $p+2$ is also a prime if and only if (2) is satisfied. *A. Brauer* (Chapel Hill, N. C.).

Toepken, Heinrich. Ein Satz über Kongruenzen. Deutsche Math. 7, 86 (1942).

A simpler proof is given of the following theorem of the author [Deutsche Math. 2, 631–633 (1937)]. Let $a(x)$ and $b(x)$ be polynomials with roots in an integral domain with characteristic 0, such that $a(x) \equiv b(x) \pmod{p^\nu}$ for a prime p and a positive rational integer ν. If $A(x)$ and $B(x)$ are the polynomials whose roots are the pth powers of the roots of $a(x)$ and $b(x)$, then $A(x) \equiv B(x) \pmod{p^{\nu+1}}$. *I. Niven.*

Niedermeier, Franz. Zwei Erweiterungen eines Kummerschen Kriteriums für die Fermatsche Gleichung. Deutsche Math. 7, 518–519 (1944).

The author proves two extensions of an early result of Kummer [J. Reine Angew. Math. 17, 203–209 (1837)]. Let q be 3 or 5 and let p be a prime quadratic nonresidue of q; then the equation $x^{2p}+y^{2p}=z^{2p}$ has no solution in integers x, y, z for which none of x, y, z is divisible by pq.
D. H. Lehmer (Berkeley, Calif.).

Brauer, Alfred, and Ehrlich, Gertrude. On the irreducibility of certain polynomials. Bull. Amer. Math. Soc. 52, 844–856 (1946).

The authors prove the following criteria for irreducibility. If, for n integral values of x, the integral polynomial $P(x)$ of degree n has values which are different from zero and, without regard to sign, less than $(n-1)!/2^{n-1}[(n-2)/2]!$, then $P(x)$ is irreducible in the field of rational numbers. If for l integral values of x, where $n>l>n/2$, the integral polynomial $P(x)$ of degree n takes values which are different from zero and, without regard to sign, less than $[(l+1)/2]$ for $l \geqq 7$ and less than $3/2$ for $l=6$ and 5, then $P(x)$ is irreducible in the field of rational numbers. The theorems are improvements on similar results of G. Pólya [Jber. Deutsch. Math. Verein. 28, 31–40 (1919)] and T. Tatuzawa [Proc. Imp. Acad. Tokyo 15, 253–254 (1939); these Rev. 1, 194]. *T. Nagell* (Uppsala).

Mahler, Kurt. Lattice points in two-dimensional star domains. II. Proc. London Math. Soc. (2) 49, 158–167 (1946).

This paper applies methods developed in the first paper of the series [same vol., 128–157 (1946); these Rev. 8, 12] to determine the constant $\Delta(H)$ and to find all the critical lattices for three simple, convex, nonsymmetric domains H which contain the origin O. The three domains considered, the ellipse, the parallelogram, and the triangle, are transformed into a circle, a square and an isosceles right-angled triangle, respectively, by suitable affinities. Each domain H is then replaced by a symmetric domain K consisting of all the points of H and of all the points of the reflection of H with respect to O. As the critical lattices of H and K are the same, $\Delta(H)=\Delta(K)$. The nonexistence of singular lattices is established easily and the results for the first two domains are then obtained by finding the parallelograms of minimum area with a vertex at O and three vertices simultaneously on the boundary of the domain H and the domain K. The third result is a consequence of Minkowski's linear form theorem. *D. Derry* (Vancouver, B. C.).

Mahler, Kurt. Lattice points in two-dimensional star domains. III. Proc. London Math. Soc. (2) 49, 168–183 (1946).

This paper, like the one reviewed above, applies the methods the author has developed in the first paper of this series to find the critical lattices and the constant $\Delta(K)$ for a special star domain K. Here K is defined to be the set of points which are within at least one of the ellipses $a_1x^2+2b_1xy+c_1y^2=1$, $a_2x^2+2b_2xy+c_2y^2=1$, $a_1c_1-b_1^2=1$, $a_2c_2-b_2^2=1$. By means of an affinity which maps one of the ellipses into a circle it is shown that $\Delta(K)$ is completely determined by the simultaneous invariant $J=a_1c_2-2b_1b_2+c_1a_2$ and so may be regarded as a function $D(J)$ of J. A simple proof is given that every critical lattice Λ has determinant $d(\Lambda) \geqq \sqrt{3}$ and contains a pair of lattice points on each ellipse, both of which pairs form a basis of Λ. The author shows that for each J there are only a finite number of lattices satisfying the two conditions above. He constructs all such lattices for $2 \leqq J \leqq 25$ and tabulates them. By use of the table the actual value of $D(J)$ is found within the interval considered. This part of the paper involves considerable detail. Upper and lower bounds for $D(J)$ for all J are stated without proof. *D. Derry* (Vancouver, B. C.).

Mahler, K. On lattice points in n-dimensional star bodies. I. Existence theorems. Proc. Roy. Soc. London. Ser. A. 187, 151–187 (1946).

Let $F(x_1, \cdots, x_n)=F(X)$ be a continuous nonnegative function over real Euclidean n-space which does not vanish everywhere and with the property that $F(tX)=|t|F(X)$ for all real t. The set of points X for which $F(X) \leqq 1$ is a star body K. A lattice Λ whose only point interior to K is the origin is defined to be K-admissible; $\Delta(K)$ denotes the lower bound of the determinants of all K-admissible lattices. A lattice with determinant $\Delta(K)$ is defined to be critical. The purpose of this paper is to study critical lattices of a star body K and especially the points of such lattices which are on or near the boundary of K. A number of fundamental results are first derived. The author introduces a definition for a bounded set of lattices and a definition for the convergence of a lattice sequence which enable him to establish a lattice point adaptation of the Bolzano-Weierstrass theorem. Much use of this is made throughout the paper. Examples of star bodies are given with finite and with infinite $\Delta(K)$. Every critical lattice of a bounded star body K is shown to contain n linearly independent points on the boundary of K. Examples are given (1) of a bounded K which has exactly one critical lattice whose only points on the boundary of K are n linearly independent points and their reflections with respect to the origin and (2) of an unbounded star body K with a critical lattice without points on the boundary of K. For every positive ϵ a critical lattice is shown to contain a point P with $1 \leqq F(P) < 1+\epsilon$. These results lead to the problem: does every critical lattice of a star body K, for all $\epsilon>0$, contain n linearly independent points P for all of which $1 \leqq F(P) < 1+\epsilon$? Most of the results of the paper contribute to partial solutions of this problem.

Let Γ be a group of linear transformations of Euclidean n-space which map the star body K into itself, such that for a given positive c and a given point X of K an element Ω of Γ always exists with $|\Omega X| < c$. If K admits such a group it is defined to be automorphic. Such star bodies are shown to possess critical lattices which contain at least one boundary point of K. If a body K has a group Γ so that for each positive c and each element $X \neq 0$ of K an element Ω of Γ exists with $|\Omega X| > c$ then infinitely many points P exist in every critical lattice so that $1 \leqq F(P) < 1+\epsilon$ for each positive ϵ. A set of conditions is given which if imposed on Γ lead to results regarding the minimum number of linearly independent points P of a critical lattice with $1 \leqq F(P) < 1+\epsilon$

for each positive ϵ. From this follows the affirmative answer for the above problem if K is defined by $F(x_1, \cdots, x_n) = (x_1 \cdots x_n)^{1/n}$. *D. Derry* (Vancouver, B. C.).

Veress, Paul. Graphische Lösung von Diophantischen Gleichungen. Mat. Fiz. Lapok **48**, 393–397 (1941). (Hungarian. German summary)

Es wird durch einfache elementargeometrische Überlegungen folgender bekannte Satz bewiesen. Ein Gitterparallelogramm hat dann und nur dann einen Flächeninhalt gleich 1, wenn auf dem Gitterparallelogramm (Begrenzung mit inbegriffen) ausser seinen Eckpunkten keine weiteren Gitterpunkte vorhanden sind. Aus diesem Satz werden die bekannten Sätze über die Diophantische Gleichung $ay-bx=1$ auf geometrischem Wege hergeleitet und eine graphische Methode zur Auffindung der Grundlösung von dieser Gleichung angegeben. *Author's summary.*

Rédei, L. Über Gitterparallelogramme. Mat. Fiz. Lapok **49**, 73–75 (1942). (Hungarian. German summary)

In der mit einem rechtwinkligen Koordinatensystem versehenen Ebene heissen die Punkte mit ganz rationalen Koordinaten Gitterpunkte, die Parallelogramme mit solchen Ecken Gitterparallelogramme. Zählt man zu diesen zwei benachbarte Seiten nicht mit, so gilt bekanntlich, dass die Anzahl der enthaltenen Gitterpunkte dem Inhalt gleich ist. Neben dem üblichen Beweis mit Grenzübergang und zwei neuerdings veröffentlichte elementare Beweisen werden zwei weitere elementare Beweise hingestellt.

From the author's summary.

Hajós, G. Über Gitterparallelogrammen. Mat. Fiz. Lapok **48**, 398–400 (1941). (Hungarian. German summary)

The author gives a simple geometrical proof of the well-known theorem that a triangle whose vertices have integer coordinates and which has no lattice point in its interior or boundary has area $\frac{1}{2}$. *P. Erdös* (Syracuse, N. Y.).

Makai, E. Über Gitterdreiecke und Gitterparallelogramme. Mat. Fiz. Lapok **50**, 47–50 (1943). (Hungarian. German summary)

Folgender bekannte Satz wird elementargeometrisch bewiesen: der Inhalt eines jeden elementaren Gitterdreiecks ist $\frac{1}{2}$. Der Beweis beruht auf folgenden beiden Tatsachen: (1) ein jedes elementares Gitterdreieck ist entweder kongruent mit dem Dreieck (0, 0), (1, 0), (1, 1), oder es hat einen stumpfen Winkel und (2) zu einem jeden stumpfwinkligen elementaren Gitterdreieck $A_1A_2A_3$ kann man ein anderes elementares Gitterdreieck $A_1A_3A_4$ mit demselben Inhalt, aber von kleinerem Umfang konstruieren.

From the author's summary.

Olds, C. D. Note on an asymmetric Diophantine approximation. Bull. Amer. Math. Soc. **52**, 261–263 (1946).

By means of geometric methods B. Segre [Duke Math. J. **12**, 337–365 (1945); these Rev. **6**, 258] deduced the following theorem on "asymmetric" Diophantine approximation. Let τ be any nonnegative number. Then, for any irrational number θ, infinitely many rationals x/y exist, such that

$$\frac{-1}{y^2(1+4\tau)^{\frac{1}{2}}} < \frac{x}{y} - \theta < \frac{\tau}{y^2(1+4\tau)^{\frac{1}{2}}}, \qquad y>0.$$

The author gives a very short and purely arithmetical proof for the special case $\tau \geqq 1$. *J. Popken* (Groningen).

Tietze, Heinrich. Systeme von Partitionen und Gitterpunktfiguren. II. Komprimierte Gitterpunktmengen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1940**, 69–131 (3 plates) (1940).

Tietze, Heinrich. Systeme von Partitionen und Gitterpunktfiguren. III. Ein Satz über das Verhältnis der Lösungsanzahlen gewisser Partitionsaufgaben. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1940**, 133–145 (1940).

Tietze, Heinrich. Systeme von Partitionen und Gitterpunktfiguren. IV. Formeln und Tabellen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1940**, 147–166 (1940).

Tietze, Heinrich. Systeme von Partitionen und Gitterpunktfiguren. V. Der Hauptsatz über den Umbau komprimierter Gitterpunktmengen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1941**, 1–37 (1941).

Tietze, Heinrich. Systeme von Partitionen und Gitterpunktfiguren. VI. Konvexe Polygonzüge und Partitionen nebst deren Ordnungsbeziehungen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1941**, 39–55 (1941).

Tietze, Heinrich. Systeme von Partitionen und Gitterpunktfiguren. VII. Schrittweise Kompression partiellkomprimierter Mengen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1941**, 165–170 (1941).

Tietze, Heinrich. Systeme von Partitionen und Gitterpunktfiguren. VIII. Auswirkung der Kompression von Gitterpunktmengen auf die zugehörigen Partitionen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1941**, 171–186 (1941).

Tietze, Heinrich. Systeme von Partitionen und Gitterpunktfiguren. IX. Beispiele ähnlich geordneter Familien von komprimierten Gitterpunktmengen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1941**, 187–191 (1941).

The first of this series of papers on systems of partitions and their corresponding lattice points appeared in the same S.-B. **1940**, 23–54; these Rev. **3**, 166. It contains a general statement of the problem, which, to put it roughly, is to study the relationships between sets of $n+1$ partitions of m

$$a^{(\nu)} = (a_1^{(\nu)}, \cdots, a_m^{(\nu)}), \qquad \nu = 0, 1, \cdots, n,$$

into m nonnegative parts, and geometric realizations of such partitions by "graphs," or lattice point sets in $(n+1)$-dimensional space. Paper I discussed various recursion formulas for the number of such graphs that may be made to correspond with a given set of partitions.

Paper II deals with the $(n+1)$-dimensional analogue of the notion of "regular" or "reduced" Ferrers graph [Tietze, Math. Z. **47**, 352–356 (1941); these Rev. **4**, 36]. Complicated recursion formulas for the number of such graphs are deduced by elementary reasoning.

In paper III the author shows that, if the problem mentioned in paper I is soluble in N ways with a given set of $n+1$ partitions of m, then the problem obtained by adjoining another partition to the set is also possible, the number of solutions being at least $(1+1/[\frac{1}{2}m])N$. The number N is at least $(m!)^n$.

Paper IV gives examples, recursion formulas and tables for $m \leqq 5$ and $n=2$, as well as the number $p^{(3)}(m)$ of reduced graphs of m lattice points in three dimensions for $m \leqq 14$.

Paper V defines a class of transformed lattice point sets arising from a reduced lattice point set. This class is built up of elementary transformations in which only two coordinates are changed at a time. Illustrations are given.

Paper VI deals with the problem of ordinary partitions and its connection with certain polygonal lines. To the partition $(a_1, a_2, \cdots, a_m)$ of the integer m into nonnegative parts a_ν arranged in nonincreasing order one may make correspond a unique polygonal line passing through the points whose coordinates are $(0, 0)$ $(1, a_1)$, $(2, a_1+a_2)$, $(3, a_1+a_2+a_3)$, $\cdots$, (m, m). If one polygonal line P_1 lies above another P_2 we say that $P_1 > P_2$ and the corresponding partitions are ordered in the same way.

Paper VII introduces the notion of a partially reduced set of lattice points. A set M of lattice points in n-dimensional space $(x_1, \cdots, x_n)$ is said to be reduced with respect to $x_1, \cdots, x_k$ in case it has the property that if $(x_1', \cdots, x_n')$ is a member of M so also is every point $(x_1, \cdots, x_n)$ for which

$$0 < x_\nu < x_\nu', \qquad \nu = 1, 2, \cdots, k,$$
$$x_\nu = x_\nu', \qquad \nu = k+1, \cdots, n.$$

If $k=n$ we have the case of a completely reduced set of points. A process called "x compression" is defined, the successive application of which transforms a partially reduced set to a totally reduced set. This notion is used in paper VIII, where it is shown that x compression raises the order of the corresponding partition in the sense of paper VI.

Paper IX gives further illustrations of the ideas in paper V.

D. H. Lehmer (Berkeley, Calif.).

Jarník, Vojtěch. Über die Mittelwertsätze der Gitterpunktlehre. VI. Časopis Pěst. Mat. Fys. **70**, 89–103 (1941). (Czech. German summary)

Let $Q(u) = \sum_{j=1}^{r} \alpha_j u_j^2$ $(\alpha_j > 0)$ and let V be the volume of the ellipsoid $Q(u) \leqq 1$, $A(x)$ the number of lattice points $(u_1, \cdots, u_r)$ satisfying $Q(u) \leqq x$, and

$$M(x) = \int_0^x |A(y) - Vy^{\frac{1}{2}r}|^2 dy.$$

By methods similar to those used in part V [same Časopis **69**, 148–174 (1940); these Rev. **3**, 274] the author shows that $M(x) = O(x^{\frac{1}{2}})$ $(r=2)$ and $M(x) = O(x^2 \log x)$ $(r=3)$, where the α_j are arbitrary positive numbers, rational or irrational. These supplement the known results for $r>3$ and supersede the author's earlier estimates which were $O(x^{\frac{1}{2}} \log^4 x)$ and $O(x^2 \log^4 x)$, respectively [Math. Z. **33**, 62–84 (1931)].

R. A. Rankin (Cambridge, England).

Niven, Ivan. A note on the number theory of quaternions. Duke Math. J. **13**, 397–400 (1946).

It is shown that every Hurwitz integral quaternion is expressible as a sum of three squares of (Hurwitz) integral quaternions, that a Lipschitz integral quaternion is expressible as a sum of three squares of (Lipschitz) integral quaternions if and only if it has the form $a+2bi+2cj+2dk$, and that these results are best possible. *T. Estermann.*

Reitan, L. A solution of the equation $am^2 = x^2+y^2+z^2$. Norsk Mat. Tidsskr. **24**, 43–47 (1942). (Norwegian)

Molsen, Karl. Zur Verallgemeinerung des Bertrandschen Postulates. Deutsche Math. **6**, 248–256 (1941).

The author gives elementary proofs of two generalizations of "Bertrand's postulate" that between n and $2n$ there is always at least one prime. He proves that (1) for $n \geqq 199$, the interval $n < p \leqq \frac{9}{8}n$ always contains a prime of each of the forms $3x+1$, $3x-1$; (2) for $n \geqq 118$, the interval $n < p \leqq \frac{9}{8}n$ always contains a prime of each of the forms $12x+1$, $12x-1$, $12x+5$, $12x-5$. *H. Davenport.*

Brun, Viggo. The integral-logarithm as an expression for the number of prime numbers less than x. Norsk Mat. Tidsskr. **26**, 41–50 (1944). (Norwegian)

Tchudakoff, N. On zeros of Dirichlet's L-functions. Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 47–56 (1946). (Russian. English summary)

The main results are the following two theorems. (1) Dirichlet's function $L(s, \chi)$, where χ is a character mod k $(\geqq 3)$, has no zeros in the part of the s-plane defined by

$$0 < |t| \leqq t_0, \quad \sigma \geqq 1 - c(\log k)^{-1};$$
$$|t| \geqq t_0, \quad \sigma \geqq 1 - c(\log |t|)^{-\gamma},$$

where $s = \sigma + it$, c and γ are positive absolute constants, $\gamma < 1$, and $\log t_0 = (\log k)^{1/\gamma}$. [The need for the condition $|t| > 0$ arises from the possibility that (so far as is known at present) one L-function belonging to a given k, say $L(s, \chi_1)$, may have one real simple zero β_1 between 1 and $1 - c(\log k)^{-1}$.]

(2) Let $\pi(x, k, l)$ be the number of primes p for which $p \leqq x$, $p \equiv l \pmod{k}$, where $(l, k) = 1$. Then, if $1 \leqq k \leqq \exp(\log x)^{\gamma}$,

$$\pi(x, k, l) = k^{-1} \sum_{n=2}^{x} \frac{1}{\log n} - Ek^{-1}\chi_1(l) \sum_{n=2}^{x} \frac{n^{\beta_1 - 1}}{\log n} + O(xe^{-c'(\log x)^{\mu}}),$$

where $h = \varphi(k)$, γ has the same meaning as in (1), $\mu = (1+\gamma)^{-1} > \frac{1}{2}$, c' is a positive absolute constant, $E = 1$ or 0 according as the special zero β_1 does or does not exist (in the position stated), and O implies an absolute constant. [In the English summary the restriction on k is omitted.]

The essential novelty lies in a combination of the precision afforded by Vinogradov's estimations of trigonometrical sums with an explicit specification of the dependence on k based on earlier work of Landau, Titchmarsh and Page. With the help of the researches of Heilbronn and Siegel [C. L. Siegel, Acta Arith. **1**, 83–86 (1935)] it is shown that the terms involving β_1 in (2) can certainly be omitted if $k \leqq (\log x)^A$, provided that the constant of the O is now understood to depend on the constant A.

The task of following out the intricacies of the argument is made needlessly difficult by a number of inaccuracies of detail. Obvious "misprints" will be easily detected, and the missing error terms in the integral formula for $S(n)$ on page 54 can be supplied by reference to Landau's paper. But lemma 1 is more puzzling. This is quoted as a known result of Vinogradov, but is not easy to identify from the references given, and is plainly false as stated. There is in any case no direct reference to lemma 1 in the course of the paper. The appeal to the relevant result of Vinogradov comes in lemma 4, through a paper by Tchudakoff. The reference to that paper should, however, be completed by an appeal to lemma 3 as well as to lemmas 1 and 2.

A. E. Ingham (Cambridge, England).

★Patz, Wilhelm. Tafel der regelmässigen Kettenbrüche für die Quadratwurzeln aus den natürlichen Zahlen von 1–10000. Akademische Verlagsgesellschaft, Leipzig, 1941; J. W. Edwards, Ann Arbor, Mich., 1946. xiv+282 pp. $6.50.

Let

$$D = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cdots}} \tag{1}$$

be the regular continued fraction for the square root of the integer D. As is well known, the b's, beginning with b_1, are

periodic and if the period is p then $b_p=2b_0$. Moreover, $b_k=b_{p-k}$ $(k>0)$ so that the b's in a period are palindromic.

This table gives for each $D\leqq 10002$ the value of b_0 and the first half of the periodic b's including $b_{p/2}$, when p is even, which is marked with an asterisk, and finally b_p. If p is odd, a small diamond shaped sign is placed before the expansion. When D is a prime, a small p is placed after it in the argument column. The nth complete quotient of the continued fraction (1) may be put in the form $x_n=(\sqrt{D}+P_n)/Q_n$. For many applications the Q_n are needed and they have been given in previous tables of lesser extent. The inclusion of these useful numbers would have doubled the size of the table. As Perron points out [J. Reine Angew. Math. **185**, 106–110 (1943); these Rev. **5**, 91] this defect can be remedied to some extent in the case of small Q_n. For example, if $Q_n=4$, then b_n is either $[b_0/2]$ or $[b_0/2]-1$. This table will be useful in studying many questions which are still unanswered concerning the expansion itself and related topics in quadratic fields and binary quadratic forms. *D. H. Lehmer.*

Guinand, A. P. Integral modular forms and summation formulae. Proc. Cambridge Philos. Soc. **43**, 127–129 (1947).

Formulas concerning sums of the coefficients of modular forms are obtained from a more general theorem proved earlier [Quart. J. Math., Oxford Ser. **9**, 53–67 (1938)].

H. S. Zuckerman (Seattle, Wash.).

Brandt, H. Zur Zahlentheorie der Quaternionen. Jber. Deutsch. Math. Verein. **53**, 23–57 (1943).

It has been known for many years that the arithmetic of totally definite quaternion algebras, that is, normal division algebras of degree 2 over totally real fields with definite norm forms, is much more complicated than that of other normal simple algebras over algebraic fields. This arithmetic has been associated in various ways with quadratic and with Hermitian forms. Also, the splitting field theory of simple algebras, their general ideal theory, of which Brandt's early work on quaternions was the forerunner, and p-adic methods have been employed. Brandt has largely based his studies on his composition theory of quaternary quadratic forms, but familiarity with that theory is not necessary for an understanding of the present paper, which is nearly complete in itself, and in which are many new as well as known results. The paper deals chiefly with the rational case, but the author states that some of his discussion could be carried out more generally.

He first establishes a simple and direct correspondence between quaternion orders and primitive ternary quadratic forms, which is such that two ternary forms determine the same algebra if and only if a multiple of one is rationally transformable into the other, and they determine the same order if and only if they are equivalent. To get all quaternion orders, in particular, maximal orders, one must employ ternary forms f with odd coefficients in the product terms, as well as classical forms. From the coefficients of f and of the adjoint of $2f$ are easily determined the coefficients of the norm form of the corresponding order and the multiplication constants of an order basis having 1 as its first element. The fundamental number d of $\mathfrak{Q}$ can be determined from f, as well as the discriminant c^4d^3 of the order determined by f.

Secondly, the author introduces ideal equivalences for regular ideals $\mathfrak{a}$, $\mathfrak{b}$, $\cdots$, belonging to maximal orders $\mathfrak{o}$, $\mathfrak{o}'$, $\cdots$. He takes $d<0$ in the definite case, as is now usual. Then d is the negative product of an odd number r of distinct primes. The 2^r positive factors of d form the divisor group $T(\mathfrak{Q})$ of $\mathfrak{Q}$, under multiplication with the discarding of square factors. Ideals $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent if $\mathfrak{b}=\rho\mathfrak{a}\mathfrak{g}\sigma$, with ρ and σ in $\mathfrak{Q}$ and $\mathfrak{g}=\mathfrak{o}\mathfrak{g}=\mathfrak{g}\mathfrak{o}$ a primitive two-sided ideal belonging to the right order $\mathfrak{o}$ of $\mathfrak{a}$. In particular, if $\mathfrak{g}=\mathfrak{o}$ and ρ and σ are arbitrary, one has ordinary or natural equivalence and likewise for the classes; if T is any subgroup of $T(\mathfrak{Q})$, $N(\mathfrak{g})\epsilon T$, and no prime factor p of d occurs in $N(\rho)$ or $N(\sigma)$ to an odd power, positive or negative, for $p\epsilon T$, one has T-equivalence. If $T=1$ and $T=T(\mathfrak{Q})$, T-equivalence is prime equivalence and wide equivalence, respectively, giving corresponding classes. If $\sigma=1$ or $\rho=1$, one has the various right- and left-equivalences, respectively. For any equivalence, a unit class is one which contains a maximal order. The natural classes correspond to equivalence of norm forms, but as a rule they do not have a simple composition yielding a groupoid, but form instead a "gestörte Gruppoide." The prime classes do compose to form a groupoid whose class k, that is, the order of the subgroups belonging to each of its units, and rank h, that is, the number of units, the author calculates. His calculations involve the orders $l_0=l_0(\mathfrak{o})$, and indices j_0 in $T(\mathfrak{Q})$, of subgroups of $T(\mathfrak{Q})$, which may be different for different orders, and may be defined as the set $T(\mathfrak{o})$ of factors of d which are represented by the norm form of $\mathfrak{o}$. Formulas of H. J. S. Smith, and of Smith and Eisenstein, for weights of genera of quaternary and ternary genera, respectively, are employed. The author uses his composition table [Math. Ann. **96**, 360–366 (1926)] for the groupoid of prime classes to exhibit and clarify composition relations in a simple way. Let the $n=kh$ elements of the groupoid which have the same left unit S_1 be $S_1, S_2, \cdots, S_n$ and arrange the n^2 products $S_a^{-1}S_b=S_{a,b}$ in a square array with $S_{a,b}$ in the ath row and bth column, $a, b=1, \cdots, n$. Then $S_{a,b}S_{b,c}=S_{a,c}$ and all compositions possible in the groupoid are so obtained. By a suitable numbering, determined by the natural classes to which the S's belong, the table can be so arranged, preserving the composition property, that a grouping of the array into k by k squares in h rows and h columns will give the complexes of prime classes which constitute the natural classes. Each complex $\mathfrak{S}_{d,e}$, $d, e=1, \cdots, h$, will have each of its distinct elements occurring in it equally often, but the numbers will be different, in general, for different complexes. The complexes on the diagonal have $j_0^2l_0$ distinct elements each occurring l_0 times, and are groupoid tables for the prime classes forming the unit natural classes, where j_0 and l_0 may vary along the diagonal subject to $j_0l_0=k$. With a "composition" $\mathfrak{S}_{d,e}\mathfrak{S}_{e,f}\supset\mathfrak{S}_{d,f}$, the complexes form a gestörte Gruppoide. The author illustrates with values of d the variety possible here.

Finally, the author gives a new derivation, and proof of the commutativity, of what he calls the Hecke matrices of $\mathfrak{Q}$ [Hecke, Math. Ann. **114**, 1–28, 316–351 (1937)]. The discussion is simpler than his earlier one [Math. Ann. **117**, 758–763 (1941); these Rev. **3**, 164] and he first illustrates it in the simpler, commutative case of a quadratic field of discriminant Δ. Suppose the set (m) of distinct ideals of the field of norm m is nonvacuous, and is distributed in the composition table of the ideal class group according to classes, thus giving a matrix $\{m\}$ which has sets of ideals as its elements, including, possibly, empty sets. Each row of $\{m\}$ will contain the whole (m). If m' is prime to m, the matrix product $\{m\}\{m'\}=\{mm'\}$, with set-theoretic interpretation of sums. It is also shown that, if $(\Delta/p)=1$,

$\{p^\mu\}\{p^{\mu'}\}=\{p^{\mu+\mu'}\}+p\{p^{\mu+\mu'-2}\}+\cdots+p^\rho\{p^{\mu+\mu'-2\rho}\}$, where ρ is the largest integer such that $\mu+\mu'-2\rho\geqq 0$. When $(\Delta/p)=0$ or -1, $\{p^\mu\}\{p^{\mu'}\}=\{p^{\mu+\mu'}\}$. The Hecke matrices $\lambda(m)$ are obtained by replacing in $\{m\}$ the sets of ideals by the numbers of ideals in the sets. They satisfy $\lambda(m)\lambda(m')=\sum\lambda(mm'/t^2)$, summed over the common divisors t of m and m' which contain only prime factors p for which $(\Delta/p)=1$. The difficulties in the quaternion case due to restricted multiplication and to the existence of an infinity of ideals of norm m are overcome by using the Verschiebungsprodukt: $\mathfrak{a}\leftarrow\mathfrak{b}=\mathfrak{c}$, defined only if $\rho^{-1}\mathfrak{o}'\rho=\mathfrak{o}$, where $\mathfrak{a}\mathfrak{o}=\mathfrak{a}$, $\mathfrak{o}'\mathfrak{b}=\mathfrak{b}$, as $\mathfrak{c}=\mathfrak{a}\rho^{-1}\mathfrak{b}\rho$, and similarly for complexes, and by making an appropriate selection of ideals as sets (m). With essentially the same definition for $\lambda(m)$ in this case, the author proves that $\lambda(m)\lambda(m')=\sum\Omega\lambda(mm'/t^2)$, summed for all common divisors t of m and m' which are prime to d. *R. Hull.*

ANALYSIS

Usai, Giuseppe. Alcune considerazioni sulle medie. Boll. Accad. Gioenia Sci. Nat. Catania (3) **14**, 9–15 (1940).

The author discusses elementary properties of arithmetic, geometric and harmonic means. *R. P. Boas, Jr.*

Stubban, John Olav. On the arithmetic and geometric means. Norsk Mat. Tidsskr. **26**, 116–117 (1944). (Norwegian)

Selberg, Henrik L. On an inequality in mathematical statistics. Norsk Mat. Tidsskr. **24**, 1–12 (1942). (Norwegian)

Let $f(x)\geqq 0$ and put

$$M_n=\int_{-\infty}^{\infty}|x|^n f(x)dx,\quad m_k=\int_0^{\infty}x^k f(x)dx-\int_{-\infty}^{0}|x|^k f(x)dx.$$

Suppose that $M_0=1$ and that $f(x)$ has only one maximum. Let

$$Q(y,\rho)=\frac{(y+1)^{n+1}+1}{(n+1)y}-\rho\frac{(y+1)^{k+1}-1}{(k+1)y}$$

and let $y=y(n,k)$, $\rho=\rho(n,k)$ be positive roots of the system of equations $Q=1+\rho$, $\partial Q/\partial y=0$. Then for $n>k\geqq 0$ and for any $\xi>0$ with $|m_k|\xi^{n-k}\leqq M_n$ one has

$$\int_{-t}^{t}f(x)dx\geqq 1-\frac{M_n+\rho|m_k|\xi^{n-k}}{(1+\rho)\xi^n}.$$

Various special cases are considered. *W. Feller.*

Dugué, Daniel. Un théorème de théorie des fonctions obtenu à partir de résultats de calcul des probabilités. C. R. Acad. Sci. Paris **223**, 845 (1946).

A function is said to be absolutely monotonic in the wide sense if it has a development of the form $\sum_0^\infty a_n x^n+\sum_1^\infty b_n x^{-n}$, $a_n\geqq 0$, $b_n\geqq 0$. The author shows that, if $e^x=f_1(x)f_2(x)$ with f_k absolutely monotonic in the wide sense, then $f_k(x)=e^{a_k x}$, $0<a_k<1$, $k=1,2$. *R. P. Boas, Jr.* (Providence, R. I.).

Lelong, Pierre. Sur la définition des fonctions harmoniques d'ordre infini. C. R. Acad. Sci. Paris **223**, 372–374 (1946).

Let $f(x_1,\cdots,x_p)=f(x)$ have derivatives of all orders on the open set D of R_p. Moreover, let $(-1)^n\Delta^n f(x)\geqq 0$, $n=1,2,\cdots$, where Δ^n denotes the n times iterated Laplacian. Then, if G denotes any compact set, G in D and

$$\mu_n(G)=\int_G|\Delta^n f(x)|d\tau,$$

we have $\lim_{n\to\infty}(\mu_n(G)/(2n)!)^{1/n}=0$. As a consequence of the last relation we have also $\lim_{n\to\infty}(\mu_n'(G)/(2n)!)^{1/n}=0$, where $\mu_n'(G)$ is the maximum of $|\Delta^n f(x)|$ for x in G. The main theorem is an extension to functions of p variables of a result of D. V. Widder [Trans. Amer. Math. Soc. **51**, 387–398 (1942); these Rev. **3**, 293]. *G. Szegö.*

Soula, Jacques. Sur les fonctions d'une variable réelle qui admettent des dérivées de tous les ordres. C. R. Acad. Sci. Paris **223**, 711–712 (1946).

The author states some results giving estimates for the rate of growth of the derivatives of a function in case the zeros of the successive derivatives are assumed to be far enough apart. For example, if y_n is the zero of $f^{(n)}(x)$ closest to c, then $|y_{n+1}-c|\,|f^{(n)}(c)/n!|^{1/n}$ is bounded. [Cf. A. C. Schaeffer, Trans. Amer. Math. Soc. **54**, 278–285 (1943); these Rev. **5**, 4.] *R. P. Boas, Jr.* (Providence, R. I.).

★Bang, Thøger. Om quasi-analytiske Funktioner. [On Quasi-Analytic Functions]. Thesis, University of Copenhagen, 1946. 101 pp. (Danish)

The first part of this thesis develops the theory of classes of functions defined by inequalities satisfied by their successive derivatives on $(-\infty,\infty)$. [This work is similar to, but independent of, similar investigations of H. Cartan and S. Mandelbrojt dealing principally with finite intervals [Cartan, Sur les classes de fonctions définies par des inégalités portant sur leurs dérivées successives, Actual. Sci. Indus., no. 867, Hermann, Paris, 1940; Mandelbrojt, Rice Inst. Pamphlet **29**, no. 1 (1942); these Rev. **3**, 292].] Numerous special cases are investigated and purely real-variable proofs are given for the Bernstein inequality for derivatives and the Bohr-Favard inequality for integrals.

In the second part, criteria are given for quasianalytic classes and a simple proof is obtained for the Denjoy-Carleman theorem. It is proved that a function belonging to a quasianalytic class is determined by the sequence $\{f^{(k)}(x_k)\}$, proved that $\sum|x_k-x_{k-1}|$ converges; other results of the same character are given. The summability of the Taylor series of a quasianalytic function is discussed.

In an appendix it is shown that every continuous function can be written as the sum of two Bernstein quasianalytic functions. *R. P. Boas, Jr.* (Providence, R. I.).

Batty, Joyce S., and Walker, A. G. Non-integral functional powers. Quart. J. Math., Oxford Ser. **17**, 145–152 (1946).

A node of a function $f(x)$ defined in the closed interval (a,b) of the real variable x is an x-value for which $f(x)=x$. The function $f(x)$ is called complete in (a,b) if it is continuous and strictly increasing and $x=a$ and $x=b$ are nodes of $f(x)$. The product $f\cdot g$ of the functions $f(x)$ and $g(x)$ is defined as the function $f(g(x))$ and powers f^p with integral p are defined correspondingly. In continuation of two former papers by the second author [same J. **17**, 65–82, 83–92 (1946); these Rev. **8**, 19] powers f^λ with nonintegral λ are considered. A set P of functions is called a power set in (a,b) if (i) each $f\in P$ is complete, (ii) any two functions of P commute and (iii) every integral power of an element of P and the products of such powers are either free of nodes in the interior of (a,b) or the identity function. Since these properties continue to hold if all integral powers of

all elements of P are adjoined it will be assumed that P contains all these powers. As "base function" an arbitrary $f \epsilon P$ is chosen which is different from the identity function I and therefore, because of (i), either greater than I for all interior points of (a, b) or smaller than I for all these points. For convenience the first possibility is chosen. If, then, g is an arbitrary element of P, the set of all rational numbers p/n for which $g^n > f^p$ is shown to be the lower class of a Dedekind cut, and the real number λ defined by this cut is called the index of g with respect to f. The relation $g \doteq f^\lambda$ is used to indicate that in a given power set λ is the index of g with respect to f. The authors prove some properties of the index analogous to those of ordinary exponents. In particular, it is shown that, if f, g, h are elements of P and if $g \doteq f^\lambda$, $h \doteq f^\mu$, then (i) $g^p \cdot h^q \doteq f^{p\lambda+q\mu}$ for any integers p, q, and (ii) $g > h$ or $g < h$ according as $\lambda > \mu$ or $\lambda < \mu$. The set of all power sets P is divided into three classes: (i) the integral power sets each containing only integral powers of one of its elements, (ii) rational power sets (the set of all indices is an everywhere dense set of rational numbers), (iii) irrational power sets (the set of indices contains at least one irrational number, and therefore an everywhere dense set of irrational numbers). The authors discuss the relation of this classification to the "related" and "semi-related" sets introduced in the two papers quoted above. *E. H. Rothe.*

Calculus

Riordan, John. Derivatives of composite functions. Bull. Amer. Math. Soc. **52**, 664–667 (1946).

The author establishes a close relation of the Y polynomials of E. T. Bell [Ann. of Math. (2) **35**, 258–277 (1934), in particular, p. 269] to well-known formulae for the nth derivative of a composite function $F[f(x)]$, especially to di Bruno's formula and to the more general formulae in the case of a function of many functions. *J. F. Koksma.*

Mohr, Ernst. Bemerkung zum Mittelwertsatz der Differentialrechnung. Deutsche Math. **7**, 248–251 (1943).

Lagrange's interpolation formula is used to prove the Cauchy mean value theorem and H. A. Schwarz's generalization involving determinants. *P. Franklin.*

Jucci, Ennio. Sopra una dimostrazione intuitiva della formula di Bernoulli-Taylor. Atti Mem. Accad. Sci. Padova. Mem. Cl. Sci. Fis.-Mat. (N.S.) **57**, 129–137 (1941).

Taylor's series is obtained by a formal geometrical argument. *R. P. Boas, Jr.* (Providence, R. I.).

Obreschkoff, Nicola. Sulle medie aritmetiche della serie di Taylor. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 398–403 (1943).

The author has previously given a formula for the kth Cesàro means of the partial sums of a Taylor series [Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1940**, no. 4; these Rev. **2**, 284; cf. also G. Kowalewski, Deutsche Math. **6**, 349–351 (1942); these Rev. **4**, 273]. Here he gives an alternative proof. As a special case, he obtains a simple derivation of the formula

$$f(a+h) = f(a) + \sum_{\nu=1}^{n} \{\binom{\lambda}{\nu} / \binom{2n}{\nu}\} \frac{h^\nu}{\nu!} \{f^{(\nu)}(a) + (-1)^{\nu-1} f^{(\nu)}(a+h)\} + R_n,$$

$$R_n = \frac{(-1)^n}{(2n)!} \int_a^{a+h} (a+h-t)^n (t-a)^n f^{(2n+1)}(t)\,dt,$$

previously discussed (with a different expression for R_n) by K. Petr [Časopis Pest. Mat. Fys. **44**, 454–455 (1915)], G. N. Watson [ibid. **65**, 1–7 (1935)] and G. Ricci [Ann. Mat. Pura Appl. (4) **15**, 187–196 (1936)].

R. P. Boas, Jr. (Providence, R. I.).

Dufresnoy, Jacques. Sur un théorème de M. Obrechkoff. C. R. Acad. Sci. Paris **223**, 773–775 (1946).

The author shows that, if $f(x)$ has n derivatives for $x > x_0$ and $\int^\infty t^{n-1} f^{(n)}(t)\,dt$ converges, then

$$f(x) = \sum_{k=0}^{n-1} a_k x^k + \frac{(-1)^n}{(n-1)!} \int_0^\infty u^{n-1} f^{(n)}(u+x)\,dx,$$

where the a_k are constants and the integral approaches zero as $x \to \infty$. As a consequence he deduces a result of Obrechkoff [same C. R. **223**, 370–372 (1946); these Rev. **8**, 139].

R. P. Boas, Jr. (Providence, R. I.).

Grüss, G. Anormale Extremwerte von Funktionen einer Veränderlichen. Jber. Deutsch. Math. Verein. **53**, 6-9 (italic) (1943).

Examples of $f(x)$ having a minimum at $x = c$ are studied which are anormal in that although $f'(x)$ exists near c, it is not of fixed sign in any intervals $(c-h, c)$ or $(c, c+h)$.

P. Franklin (Cambridge, Mass.).

Hamel, Georg. Direkte Ableitung der Stirlingschen Formel aus dem Eulerschen Integral. Deutsche Math. **6**, 277–281 (1941).

The author gives an elementary derivation of an asymptotic series for $z^{-z}e^z\Gamma(z)$, including an estimate for the error; the first term gives Stirling's formula. *R. P. Boas, Jr.*

Quarleri, A. Studio su una speciale curva di interesse fisico. Atti Accad. Ligure **2**, 175–182 (1942).

There is deduced, on the basis of the formal character of the curve of magnetic hysteresis, an equation of the sixth degree which suffices to represent a line which has the shape of the generic curve of this kind.

From the author's summary.

Theory of Functions of Complex Variables

★**Tricomi, Francesco. Funzioni Analitiche.** 2d ed. Consiglio Nazionale delle Ricerche. Monografie di Matematica Applicata. Nicola Zanichelli, Bologna, 1946. vii+134 pp. 300 lire.

This brief introduction to the theory of functions of a complex variable is little changed from the edition of 1936. The fundamentals of the subject are set forth clearly, under the simplest hypotheses; many geometrical illustrations are given. The first chapter deals with the notion of analytic function, connections with harmonic functions and conformal mapping. The second chapter develops complex integration through the elements of the calculus of residues and Liouville's theorem. The third chapter discusses Taylor and Laurent series and isolated singular points. The fourth chapter deals briefly with analytic continuation, the elementary transcendental functions, and the Mittag-Leffler and Weierstrass representations of meromorphic and entire functions. *R. P. Boas, Jr.* (Providence, R. I.).

Giorgi, G. Punti di vista moderni nello studio delle funzioni analitiche. Rend. Sem. Mat. Fis. Milano **14**, 81–88 (1940).

One can study functions of a complex variable by a synthetic method which puts in evidence all those properties which are invariant under homographic transformations. The classical concept of singular points is replaced by the new concept of "absolute points," which do not include poles but do include stationary points, in addition to branch points and transcendental singular points. The method puts in evidence properties which escape the classical analytical methods and leads to a rational classification of functions.

From the author's summary.

Birkhoff, George D. Déformations analytiques et fonctions auto-équivalentes. Ann. Inst. H. Poincaré **9**, 51–122 (1939). [MF 16142]

A study is made of functions $f(x)$ of the complex variable x, near $x=\infty$, with the property that, for some $\varphi(x)$ of the form (1) $\varphi(x)=qx+\varphi_0+\varphi_1x^{-1}+\cdots$ $(q\neq 0)$ one has (2) $f(\varphi(x))=a(x)f(x)$, where $a(x)=x^k(a_0+\cdots)$ (integral k; $a_0\neq 0$). The possibility of extension of these results to square matrices $F(x)$ of order n, such that (2′) $F(\varphi(x))=A(x)F(x)$, with the elements of A meromorphic at ∞, is indicated but not carried out. The functions studied are meromorphic near $x=\infty$ and uniform on a Riemann surface R, having a branch point of infinite order at $x=\infty$; two such functions y, z for which y/z is uniform and meromorphic at ∞ are called equivalent. A deformation is an analytic one-to-one transformation of the x-plane near $x=\infty$ on itself; $f(x)$, meromorphic on R, is self-equivalent if there is a nonidentical deformation leaving f equivalent to itself (i.e., at least one equation (2) holds for f). The transformations $x'=\varphi_1$, $x'=\varphi_2, \cdots$ admitted by a self-equivalent f form a group G. Let T be any transformation $x=\psi(x^*)=rx^*+\psi_0+\cdots$, $r\neq 0$; G is then replaced by $G^*=TGT^{-1}$ and (2) by (2*): all $f^*=f(\psi)$ and groups G^* are of the same class. The importance of the class of self-equivalent functions is due to the fact that it contains the functions defined by fairly general ordinary linear differential, difference, q-difference and some other functional equations. Space does not allow us to state even the main results. We shall merely point out that the author studies deformations $x'=\varphi(x)$ of R (leading to a problem of inverse-Riemann type), self-equivalent functions and the possible groups of deformations. Most of the results are for $n=1$. It appears to the reviewer that the transition to and adequate treatment of the general case would present heavy new difficulties. *W. J. Trjitzinsky* (Urbana, Ill.).

de Groot, J. A theorem concerning analytic continuation. II. Nederl. Akad. Wetensch., Proc. **49**, 793–801 = Indagationes Math. **8**, 496–504 (1946).

For part I see the same Proc. **49**, 213–222 = Indagationes Math. **8**, 110–119 (1946); these Rev. **7**, 514. There are now added references to the extensive literature and generalisations such as: a function defined on a set E may be interpolated in the neighbourhood of each limit point of E by an analytic function if the nth root of the nth divided difference is bounded. *A. J. Macintyre* (Aberdeen).

Shen, Yu-Cheng. Interpolation to analytic functions by rational functions. Acad. Sinica Science Record **1**, 300–302 (1945).

The author states results on interpolation to functions $f(z)$ which are analytic for $|z|<1$ and such that the surface integral

$$\iint_{|z|<1} |f(z)|^2 dS$$

exists. If $\{\alpha_{ni}\}$, $i=1, \cdots, n$; $n=1, \cdots$; $|\alpha_{ni}|<1$ are preassigned points satisfying certain conditions and $f_n(z)$ is the rational function of the form $\sum_{i=1}^{n} A_{ni}(1-\alpha_{ni}z)^{-1}$ which interpolates to $f(z)$ at α_{ni}, $i=1, \cdots, n$, and if $n\prod_{i=1}^{n}|\alpha_{ni}|^2\to 0$ as $n\to\infty$, then $f_n(z)\to f(z)$ as $n\to\infty$ uniformly for $|z|\leqq r$ whenever $0<r<1$. Further results are stated, also corresponding to results of J. L. Walsh connected with line integrals, and a method of proof is suggested. *H. Kober.*

Walsh, J. L. Overconvergence, degree of convergence, and zeros of sequences of analytic functions. Duke Math. J. **13**, 195–234 (1946).

Let $V(z)$ be harmonic in the region R and $F_n(z)$ analytic in R except for branch points but $|F_n(z)|$ single-valued. If for every continuum Q in R we have

$$\limsup_{n\to\infty} \{\max |F_n(z)|, z\in Q\} = \max \{e^{V(z)}, z\in Q\},$$

we say that $V(z)$ is an exact harmonic majorant of the sequence $\{F_n(z)\}$. The main theorem on overconvergence is as follows. Let $f_n(z)$ be analytic in R and $V(z)$ an exact harmonic majorant of the sequence $\{f_{n+1}(z)-f_n(z)\}^{1/n}$ in R. Then $\lim_{n\to\infty} f_n(z)=f(z)$ exists in R_0: $V(z)<0$, and $V(z)$ is an exact harmonic majorant of $\{f(z)-f_n(z)\}^{1/n}$ in R_0 and of $\{f_n(z)\}^{1/n}$ in $R-R_0$. If on a particular continuum Q_0 in R_0 we have

$$\limsup_{k\to\infty} \{\max |f(z)-f_{n_k}(z)|^{1/n_k}, z\in Q_0\} < \max \{e^{V(z)}, z\in Q_0\},$$

or if on a particular continuum Q_0 in $R-R_0$ we have

$$\limsup_{k\to\infty} \{\max |f_{n_k}(z)|^{1/n_k}, z\in Q_0\} < \max \{e^{V(z)}, z\in Q_0\},$$

(n_k a specific sequence), then $\{f_{n_k}(z)\}$ converges uniformly in a certain neighborhood of any point α on $V(z)=0$ where $f(z)$ is analytic, provided α and Q_0 are not separated in R by a natural boundary of $f(z)$. In addition the degree of convergence and the distribution of the zeros (in relation to overconvergence) are studied. Overconvergence on special sets of positive capacity or positive superficial measure is also considered. *G. Szegő* (Stanford University, Calif.).

★Dvoretzky, Aryeh. Studies on general Dirichlet series. Summary of a thesis, Hebrew University, Jerusalem, 1941. 10+4 pp. (Hebrew. English summary)

The author states that the main results of the first three chapters of the thesis were published in C. R. Acad. Sci. Paris **206**, 970–972, 1438–1440; **207**, 1159–1161 (1940), and restricts his summary to the results of the last two chapters. The results quoted from the fourth chapter deal with multiplying the terms of a Dirichlet series by a sequence of ± 1's; some of these were given by the author in C. R. Acad. Sci. Paris **221**, 687–689 (1945); these Rev. **8**, 20. The results quoted from the last chapter include the following. Let $f(z)=\sum_{n=1}^{\infty} a_n z^n$, $|z|<1$, $0<\limsup |a_n|<\infty$. Let z_0 be a regular point of $f(z)$, $|z_0|=1$, and let d_n denote the distance from z_0 to the set of zeros of $\sum_{k=1}^{n} a_k z^k - a$. Then $\liminf (n/\log n) d_n < \infty$. *R. P. Boas, Jr.*

Ferenczi, Z. Über die Konvergenz einer Potenzreihe. Mat. Fiz. Lapok **50**, 29–33 (1943). (Hungarian. German summary)

A well-known result of Fejér is an asymptotic estimate for the coefficients a_n of the power series

$$(1-x)^{-\rho}\exp((x-1)^{-1})=\sum a_n x^n.$$

It follows from Fejér's estimate that the series $\sum a_n$ converges for $\rho<\frac{1}{2}$ and diverges for $\rho\geqq\frac{1}{2}$. A direct proof of the convergence is given here, using Fejér's integral representation of a_n. *O. Szász* (Cincinnati, Ohio).

Jabotinsky, Eri. A rapidly convergent series for the Weierstrass elliptic function. Riveon Lematematika **1**, 30–31 (1946). (Hebrew)

In the author's opinion, the Weierstrass elliptic functions are superior to the Jacobi functions in all respects but one, the exception being the convergence of the defining series: the series representing Jacobi functions converge much more rapidly than those representing Weierstrass functions. To remedy this the author considers the infinite series

$$\sum(-)^{m+n+s}(z-2m\omega_1-2n\omega_2)^{-1}\times\exp\{2(z-m\omega_1-n\omega_2)(m\eta_1+n\eta_2)\},$$

in which summation is extended over all integer m, n. He conjectures that this series is always convergent, and proves that if the series is convergent its sum is $\wp(z)\sigma(z)$ (in the usual notation).

The particular case $2\omega_1=1$, $2\omega_2=i$ (which leads to a convergent series) arises in some work on the number of lattice points in a circle. *A. Erdélyi* (Edinburgh).

Bermant, A. On a generalization of Koebe's theorem. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 379–381 (1946).

Extension of Koebe's theorem to functions $W(z)$ of the class R (regular in $|z|<1$ and normed by $W(0)=0$, $|W'(0)|=1$): no point of the circle $|W|<\frac{1}{4}$ can be joined to $W=\infty$ by any continuous path not intersecting the set T of values of $W(z)$. If $W=\omega$ is not covered by T, several limitations for $|\omega|$ are given. Extension of the classical formulae of distortion (limitations of $|W(z)|$ and $|W'(z)|$) when the function is univalent on the diameter of $|z|<1$ passing through the point z. *J. Ferrand* (Caen).

Teichmüller, Oswald. Vollständige Lösung einer Extremalaufgabe der quasikonformen Abbildung. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1941**, no. 5, 18 pp. (1941).

The author's basic paper in this field [same Abh. **1939**, no. 22; these Rev. **2**, 187] contained many unproved conjectures. This is a precise proof of the most important one, in the simplest possible case, which is that of a simply connected region with five distinguished boundary points. The existence of an extremal quasiconformal map is proved by a straightforward, but by no means trivial, application of the continuity method. *L. Ahlfors.*

Teichmüller, Oswald. Bestimmung der extremalen quasikonformen Abbildungen bei geschlossenen orientierten Riemannschen Flächen. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1943**, no. 4, 42 pp. (1943).

This paper supersedes the one reviewed above, the restriction to closed surfaces being probably inessential. It is proved that every class of topological maps of a closed surface onto another contains an extremal quasiconformal map, i.e., one for which the upper bound of the dilatation is a minimum. After a preliminary conformal mapping, determined by an everywhere regular quadratic differential, the extremal transformation amounts to a stretching in the direction of the imaginary axis. The surfaces are represented by means of Fuchsian polygons, the parameters being coefficients in a set of independent automorphisms. The proof is again based on the method of continuity. The crucial step is a continuity proof in which the author makes essential use of auxiliary quasiconformal maps. This is a method which he has used also on earlier occasions. The paper is an original contribution to the study of the space of closed Riemann surfaces of given genus. The author states, without complete proof, that this space, when properly defined, is homeomorphic with $(6g-6)$-dimensional Euclidean space. *L. Ahlfors* (Cambridge, Mass.).

Lavrentiev, M. Les représentations quasi-conformes et leurs systèmes dérivés. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 287–289 (1946).

Soient D (ou Δ) un domaine du plan $z=x+iy$ (ou $w=u+iv$). Si on peut définir une correspondance homéomorphe entre D et Δ, (1) $u=u(x, y)$, $v=v(x, y)$, u et v étant solutions du système

$$(2)\qquad \Phi_i\left(x, y, u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)=0, \qquad i=1, 2,$$

l'auteur dit que (2) admet une représentation quasi-conforme de D sur Δ. Ceci étant, l'auteur linéarise la correspondance (1) et la caractérise par quatre paramètres appropriés, fonctions des coefficients de la transformation linéarisée; ces paramètres sont liés par un système de deux relations (que nous nommerons (R)) contenant, en plus, x, y, u et v et qui peuvent remplacer (2) au point de vue qui nous occupe. Le système (2) sera dit fortement elliptique si (R) correspondant possède certaines propriétés de régularité; lorsque (2) est linéaire et homogène en $\partial u/\partial x, \cdots$, la notion introduite revient à celle des systèmes elliptiques. Dans des travaux, dont la référence exacte fait défaut, l'auteur établit le théorème d'existence pour le problème de représentation quasi-conforme moyennant certaines conditions à imposer à (R) et aux frontières de D et de Δ.

Cela posé, l'auteur associe à (2) un système linéaire de deux équations de premier ordre non-homogène qu'il nomme système dérivé d'ordre un (dont on peut, moyennant des hypothèses de régularité convenables, déduire des dérivés d'ordre quelconque) et qui sera elliptique si (2) l'est fortement. On est conduit à examiner divers cas particuliers intéressants, dont les équations de la dynamique plane des gaz, formées par Tchaplyguine. Enfin, l'auteur énonce un théorème qui relie les propriétés de (2) à celles de la correspondance (1). *J. Kravtchenko* (Grenoble).

Skrylyeff, V. The relations between the simplest symmetrical functions of the roots of an integral transcendental function and its derivative. Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 155–163 (1946). (Russian. English summary)

From the identity

$$\frac{d}{dz}\left(\frac{f'}{f}\right)=\frac{f'}{f}\left\{\frac{f''}{f'}-\frac{f'}{f}\right\}$$

the author deduces relations between the power sums $\sum_{j=1}^{\infty}a_j^{-k}$ of the zeros a_j of an integral function of finite order $f(z)$ and the corresponding sums related to $f'(z)$. Applications to special functions are given. From a new recurrence

relation thus obtained the author infers that the denominator of the Bernoulli number B_{2m} is a factor of $2^{2m+1}-2$.
W. H. J. Fuchs (Swansea).

Pfluger, A. Zur Defektrelation ganzer Funktionen endlicher Ordnung. Comment. Math. Helv. 19, 91–104 (1946).

If $f(z)$ is an integral function of finite order ρ with deficient values (including ∞) of total deficit 2, the zeros of $f'(z)$ are exceptionally rare. It is shown that $f'(z)$ falls into the class of integral functions of integer order studied by M. L. Cartwright [Proc. London Math. Soc. (2) 33, 209–224 (1931)] and the author [Comment. Math. Helv. 18, 177–203 (1946); these Rev. 7, 426] and has an asymptotic behaviour similar to that of e^{z^ρ}. With z restricted to a sequence of annuli $r_n<|z|<kr_n$, $f(ze^{i\alpha_n})$ converges to each defective value in one or more angles of opening π/ρ, the deficiency being the number of such angles divided by ρ. There are thus at most ρ deficient values not quite shown to be true asymptotic values. *A. J. Macintyre.*

Tsuji, Masatsugu. On the behaviour of an inverse function of a meromorphic function at its transcendental singular point. Proc. Imp. Acad. Tokyo 17, 414–417 (1941). [MF 14722]

Tsuji, Masatsugu. On the behaviour of an inverse function of a meromorphic function at its transcendental singular point. II. Proc. Imp. Acad. Tokyo 17, 474–475 (1941). [MF 14729]

Tsuji, Masatsugu. On the behaviour of an inverse function of a meromorphic function at its transcendental singular point. III. Proc. Imp. Acad. Tokyo 18, 132–139 (1942). [MF 14746]

Let $w=f(z)$ be meromorphic for $|z|<\infty$ and let ω be a transcendental singularity of the inverse function lying over the point $w=0$. Let Δ be the domain in the z-plane corresponding to the ρ-neighborhood U of ω on the Riemann surface F of the inverse function, let z_0 be in Δ, and let $\Delta(r)$ be the part of Δ contained in $|z-z_0|<r$. Let $\theta(r)$ be the part of the boundary of $\Delta(r)$ lying on $|z-z_0|=r$, $A(r)$ the area of the domain $F(r)$ on F corresponding to $\Delta(r)$, $L(r)$ the length of the image of $\theta(r)$, $\Lambda(r)$ the number of holes in $\Delta(r)$ and $S(r)=A(r)/\pi\rho^2$. Then $A(r)\to\infty$ and for any $\epsilon>0$ the variation of $\log r$ over the set of r's such that $L(r)>A(r)^{1+\epsilon}$ is finite.

Let $D_1, \cdots, D_q$ be simply connected domains in $|w|<\rho$. An island over D_i is a connected domain in U lying over D_i and having no boundary relative to D_i. If all simply connected islands over D_i have at least μ_i sheets, then

$$\sum_{i=1}^{q}(1-1/\mu_i)\leqq 1+\liminf_{r\to\infty}\Lambda(r)/S(r)\leqq 2.$$

In the course of the proof it is shown that

$$\limsup \Lambda(r)/S(r)\leqq 1.$$

The second paper corrects the proof that $A(r)\to\infty$. Some details of the first paper still remain unclear to the reviewer.

In the third paper it is shown that theorems analogous to Nevanlinna's fundamental theorems hold for the value distribution of f in Δ. As a consequence analogues of Borel's theorems also hold. Let $\overline{\Delta}$ be the simply connected domain obtained by adding to Δ its holes, let $\lambda(r)$ be the total length of the part of $|z-z_0|=r$ contained in $\overline{\Delta}$ and let $n(r)$ be the number of zeros of f in $\Delta(r)$. The author calls ω a quasidirect transcendental singularity if, for any choice of z_0 and ρ,

$$n(r)\leqq K(z_0,\rho)\int_{r_0}^{r}\frac{dr}{r\lambda(r)};$$

similarly for transcendental singularities lying over an arbitrary point w_0. The author states that, if f is of finite order ρ, then the number n of quasidirect transcendental singularities is at most 2ρ, if $n\geqq 2$. The proof is promised for a later paper. *P. C. Rosenbloom* (Syracuse, N. Y.).

Martinelli, Enzo. Studio di alcune questioni della teoria delle funzioni biarmoniche e delle funzioni analitiche di due variabili complesse coll'ausilio del calcolo differenziale assoluto. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 12, 143–167 (1942).

Sei Γ ein beschränkter Bereich des R_4 mit reell analytischer dreidimensionaler Randfläche Φ; sei ferner u eine auf Φ reell analytische Funktion, die dort den von Severi angegebenen Differentialbedingungen genüge derart, dass eine in ganz Γ biharmonische Funktion U existiert, welche auf Φ die vorgegebenen Randwerte u annimmt [siehe Severi, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (6) 13, 795–804 (1931)]. Der Verfasser stellt nun eine einfache explizite Integralformel auf, welche die Funktion U durch ihre Randwerte u ausdrückt. Im Zusammenhang mit diesem "Greenschen" Integral, gibt der Verfasser dann noch eine neue Form einer früher von ihm aufgestellten "Cauchyschen" Integralformel an, welche eine in Γ analytische Funktion zweier (oder auch von n) komplexen Veränderlichen durch ihre Randwerte auf Φ ausdrückt [siehe hierzu auch Martinelli, Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 9, 269–283 (1938)]. *P. Thullen* (Quito).

Harmonic Functions, Potential Theory

Iacob, Caius. On some harmonic functions of two variables. Pozitiva 1, 81–89 (1940). (Romanian. French summary)

Détermination des fonctions harmoniques de la forme $f[A(x)+B(y)]$. *Author's summary.*

Ciorănescu, Nicolas. Sur la détermination des fonctions harmoniques par leurs valeurs sur des variétés caractéristiques. Mathematica, Timişoara 20, 137–147 (1944).

L'auteur cherche à déterminer une fonction harmonique $u(x_1, \cdots, x_n)$ réelle au voisinage de l'origine en imposant que son prolongement dans le champ des x_i complexes selon le développement analytique prenne des valeurs données sur la variété $\sum x_i^2=0$. Dans le cas de deux variables on doit avoir $u(t, it)=f_1(t)+if_2(t)$ (t réel), $u(t,-it)=f_1(t)-f_2(t)$. Les fonctions f_1, f_2 doivent être analytiques et réciproquement si on se les donne tels quelconques, on peut aussitôt trouver u unique comme partie réelle d'une fonction holomorphe. Le cas beaucoup plus difficile de 3 variables est ensuite étudié, pour une donnée très générale, à l'aide d'une représentation intégrale des fonctions harmoniques analogue à celle de Whittaker. *M. Brelot* (Grenoble).

Tsuji, Masatsugu. On the Green's function. Jap. J. Math. 18, 379–383 (1942). [MF 14972]

The author proves the following theorem. Let D be a domain in the w-plane, whose boundary C consists of n rectifiable curves $C_1, \cdots, C_n$, and let $g(w, w_0)$ be the Green's function of D with respect to w_0, $h(w, w_0)$ any determination

of the conjugate function of g. Then (i) the normal derivative $\partial g/\partial \nu$ exists and is positive almost everywhere on C and is integrable on C; (ii) h is an absolutely continuous function of the arclength s on C and $\partial h/\partial s = -\partial g/\partial \nu$ almost everywhere on C; (iii) if $u(w)$ is a bounded harmonic function in D, then

$$u(w) = (2\pi)^{-1} \int_C u(s)(\partial g/\partial \nu) ds,$$

where $u(s)$ is the limit of $u(w)$ as w approaches the corresponding point on C, this limit existing almost everywhere. The proof uses the properties of the function which maps the universal covering surface of D onto the interior of the unit circle. *P. C. Rosenbloom* (Syracuse, N. Y.).

Bochner, S. Bloch's theorem for real variables. Bull. Amer. Math. Soc. **52**, 715–719 (1946).

Let $u_i = f_i(x_1, \cdots, x_n)$, $i = 1, \cdots, n$, be a continuously differentiable transformation defined in some open set containing the sphere $x_1^2 + \cdots + x_n^2$ and let $J(x_1, \cdots, x_n) = \partial(u_1, \cdots, u_n)/\partial(x_1, \cdots, x_n)$. For brevity, let $H(x_1, \cdots, x_n)$ denote the sum of the squares of the partial derivatives $\partial u_i/\partial x_j$, $i, j = 1, \cdots, n$. Let K be a constant such that $H \leqq K|J|^{2/n}$ in the domain described above and assume that $J(0, \cdots, 0) = 1$. Suppose finally that f_i, $i = 1, \cdots, n$, is harmonic. The author proves that there exists some open set S (in the sphere $x_1^2 + \cdots + x_n^2 \leqq 1$) which is mapped biuniquely onto some sphere of radius R_0, where R_0 depends only on n and K. This extension of a well-known theorem of Bloch is established in a simple manner, by using the Poisson integral and a "uniform" version of the existence theorem on implicit functions. The author adds several corollaries and notes that the assumptions could be further relaxed. *T. Radó* (Columbus, Ohio).

Brelot, Marcel. Fonctions sousharmoniques, presque sousharmoniques ou sousmédianes. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **21** (1945), 75–90 (1946).

The concepts of (i) upper semi-continuity and (ii) spherical mean are fundamental in the theory of subharmonic functions. In this paper, the author studies the role played by each of those concepts in the theory of subharmonic functions; particular emphasis is placed upon their roles in the convergence of sequences. For example, he points out that by modifying the usual definition of subharmonic function so as to replace the requirement of upper semi-continuity by summability, one can develop a theory of a function called sub-median by the author. Although this latter concept is new, most of the results in the paper are well known; however, the author's proofs are more elegant than those usually found in the literature.

M. O. Reade (Ann Arbor, Mich.).

Rellich, Franz. Über das asymptotische Verhalten der Lösungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten. Jber. Deutsch. Math. Verein. **53**, 57–65 (1943).

To state the first of the author's two principal theorems let $x_1, \cdots, x_k$ be real variables and set $r = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$. Let $u(x_1, \cdots, x_k)$ be a complex-valued function which, for $r > \rho_0$, is not identically zero, has continuous second partial derivatives and satisfies $\Delta u + \lambda u = 0$, where λ is a positive constant. Then there exists a constant $p > 0$ such that for arbitrary $\rho_1 > \rho_0$ and all sufficiently large ρ the k-fold integral of $|u^2|$ over the region $\rho_1 < r < \rho$ is not less than $p\rho$. The second theorem, of similar type, involves several hypotheses; the region $\rho_1 < r < \rho$ is replaced by a sequence of finite regions tending to an infinite region in a specialized manner. The author also discusses a number of applications of these theorems. *F. W. Perkins* (Hanover, N. H.).

Adelson-Welsky, G., et Kronrode, A. Sur le principe du maximum pour les solutions d'un système d'équations à dérivées partielles du type elliptique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 539–541 (1945).

Using the results given in the same J. **49**, 235–237 (1945) [these Rev. **8**, 142], the authors prove the following theorem. Let $u(x, y)$ and $v(x, y)$ be functions having total differentials in a simply connected domain D bounded by a simple closed curve C. Assume also that u and v satisfy (1) $\partial u/\partial x = A(x, y)\partial v/\partial y$, $\partial u/\partial y = B(x, y)\partial v/\partial x$, where A and B are real, with opposite signs, and each with fixed sign throughout G, but otherwise unrestricted. Then u and v attain their maxima and minima for $G + C$ on C. The theorem remains true if (1) is replaced by

$$\partial u/\partial x = a_{11}(x, y)\partial v/\partial x + a_{12}(x, y)\partial v/\partial y,$$
$$\partial u/\partial y = a_{21}(x, y)\partial v/\partial x + a_{22}(x, y)\partial v/\partial y,$$

with a_{12} of one sign for all points of G, and

$$4a_{12}a_{21} + (a_{22} - a_{11})^2 < 0.$$

In addition, the coefficients may depend on u, v, $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$. *A. B. Brown* (Flushing, N. Y.).

Segal, B. I. Some three-dimensional problems of the potential theory and their applications. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 323–358 (1946). (Russian. English summary)

The three-dimensional problems of potential theory considered are connected mainly with the flow of liquids through porous media. A detailed study of two important systems of wells only partly penetrating a sand is performed. The first is an infinite array of vertical wells situated in the middle of an infinite strip. It is assumed that at the vertical sides of the strip the potential is equal to zero, the upper and lower sides of the strip are impermeable and that at the well surfaces the potential has a given value. The second is an infinite array of vertical wells situated parallel to and at a given distance from one of the vertical sides of an infinite strip. This side as well as the upper and lower ones are supposed to be impermeable. Furthermore, it is supposed that at the other vertical side of the strip the potential is equal to zero and that at the well surfaces the potential has a given value. *From the author's summary.*

Marcantoni, Alessandro. Studi sul campo gravitazionale dell'ellissoide a tre assi. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **103**, 549–580 (1944).

Gavrilov, L. Solution of the direct and inverse potential-problem for the paraboloid of rotation and for the infinite parabolic cylindre (Addition). Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **10**, 499–500 (1946). (Russian. English summary)

The paper appeared in the same Bull. **9**, 521–528 (1945); these Rev. **7**, 302.

Duboshin, G. N. Expansion of reciprocal distance in the theory of potential. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 407–420 (1946). (Russian. English summary)

Let X, Y, Z be the rectangular coordinates of an "attracting" point and ρ, v, z the cylindrical coordinates

of an "attracted" point. The author gives some expansions of the reciprocal distance between these two points into double series, corresponding to $\rho>R$, $\rho<R$, where $R=(X^2+Y^2+Z^2)^{\frac{1}{2}}$, and to $\zeta=Z/R\neq 0$ or $\zeta=0$. The coefficients in these power series of the ratios z/R and ρ/R, or of z/ρ and R/ρ, respectively, are polynomials in $\gamma=(X/R)\cos v+(Y/R)\sin v$. They can also be transformed into sums of $\cos kv$ and $\sin kv$, with coefficients depending only on X, Y, Z. The author states the conditions of convergence. [There are some misprints in the final formulae.]

W. Jardetzky (Graz).

Sobrero, Luigi. Sull'integrazione del $\Delta\Delta$ in una striscia indefinita. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 182–189 (1943).

The solution of the equation $\Delta\Delta F=0$, valid in the strip $0\leqq y\leqq 1$ and such that, for $y=0$, $F=f_1$, $\partial F/\partial y_1=g_1$ and for $y=1$, $F=f_2$, $\partial F/\partial y=g_2$, f_1, g_1, f_2, g_2 being given functions of x, is deduced by a method of successive approximations. Writing

$$\frac{\partial^4 F_n}{\partial y^4}=-2\frac{\partial^4 F_{n-1}}{\partial x^2\partial y^2}-\frac{\partial^4 F_{n-1}}{\partial x^4},\qquad n>1,$$

and $F_1=a+by+cy^2+dy^3$, where a, b, c, d are so defined that F_1 satisfies the conditions for $y=0$ and $y=1$, the author proves that $F=\lim_{n\to\infty}F_n$. The function F_n-F_1 is obtained in the form

$$\int_0^1 g(\eta, y)\left\{-2\frac{\partial^4 F_{n-1}(\eta)}{\partial x^2\partial\eta^2}-\frac{\partial^4 F_{n-1}(\eta)}{\partial x^4}\right\}d\eta,$$

where $g(\eta, y)$ is the Green's function adapted to this problem. The convergence of the process is proved by obtaining an upper limit for this integral, and it appears that in the theory of elasticity the function F_1 gives the same approximation as the principle of the conservation of plane sections and that the function F_2 gives an approximation sufficient for all practical applications. [In the last formulae on page 184, giving the function g, an error is to be corrected. The formulae should be

$$g=\tfrac{1}{6}\eta^2(1-y)^2\{-y+\eta-2y(1-\eta)\},\qquad \eta\leqq y,$$
$$g=\tfrac{1}{6}y^2(1-\eta)^2\{-\eta+y-2\eta(1-y)\},\qquad \eta\geqq y.$$

On page 185 in the formulae giving h, k should be replaced by $-k$. The function g is negative for all values of η which occur, but the formulae on page 186 giving the integrals J and Y are correct, and so the proof of convergence remains valid.]

H. Bremekamp (Delft).

Tolotti, Carlo. Sulla struttura delle funzioni iperarmoniche in più variabili indipendenti. Ricerca Sci. **16**, 315–317 (1946).

The author presents without proof extensions of the classical results of Almansi concerning the decomposition of hyperharmonic functions into harmonic ones. The hyperharmonic functions $u(x_1, \cdots, x_m)$ considered are solutions of an equation $\Delta^n u=0$ in a domain D, where Δ is the Laplacian operator. The simplest decomposition would be of the form $u(P)=\sum_{k=0}^{n-1}x_m^k u_k(P)$ with suitable harmonic functions u_k. Other types of representations have been given by Picone and by Fichera [see Fichera, Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 511–523 (1942); these Rev. **6**, 176]. The author finds necessary and sufficient conditions for D guaranteeing the existence of a representation of one of the various types. He finds an unexpected dependence of the conditions on the number m of dimensions.

F. John (New York, N. Y.).

Differential Equations

Müller-Strobel, Josef. Lösung des Gleichungssystems der pendelnden Synchronmaschine. Arch. Elektrotechnik **36**, 399–417 (1942).

The author considers the case of small amplitudes and derives a system of linear differential equations with constant coefficients, which he solves using the Laplace transform. Various special cases are discussed in detail.

R. Bellman (Princeton, N. J.).

Constantinescu, G. G. On linear differential equations integrable by a generalized Laplace transform. Pozitiva **4**, 7–23 (1943); **3**, 41–56 (1943); **4**, 57–72 (1944). (Romanian)

The author discusses equations integrable by means of the transforms $\int Z(z)e^{zX(x)}dz$ and $\int Z(z)e^{yY(z)}dz$.

R. P. Boas, Jr. (Providence, R. I.).

Ritt, J. F. On the singular solutions of certain differential equations of the second order. Proc. Nat. Acad. Sci. U. S. A. **32**, 255–258 (1946).

The equation $F(y'', y, x)=0$, where F is a polynomial in y'' and y with coefficients that are analytic functions of x in some area, is assumed irreducible in a given field containing the coefficients of F. Consideration of the singular solutions of the equation leads to a study of equations of the type $y''=\sum_{i=0}^{p}a_i(x)y^{(p+i)/r}$, where p and r, $p<r$, are positive integers and where the second member is analytic in x and $y^{1/r}$ at $x=0$, $y=0$. The paper shows that $y=0$ is an envelope of two one-parameter families of solutions of this latter equation, these two families being in general distinct from each other but coincident in a special case. Furthermore, where a curve of one of these families contacts $y=0$, the curvature of the curve is zero.

W. M. Whyburn.

Tóth, F. Eine neue Annäherungsmethode für gewöhnliche Differentialgleichungen erster Ordnung und eine neue Abschätzung der erreichten Genauigkeit. Mat. Fiz. Lapok **48**, 176–192 (1941). (Hungarian. German summary)

A modification of Picard's iteration method yields a solution of the differential equation $y'=f(x, y)$ with the initial condition $y(x_0)=y_0$, assuming that $f(x, y)$ and f_y are continuous in the domain $x_0\leqq x\leqq X$, $|y-y_0|\leqq M(x-x_0)$.

O. Szász (Cincinnati, Ohio).

Lettenmeyer, F. Über die von einem Punkt ausgehenden Integralkurven einer Differentialgleichung zweiter Ordnung. Deutsche Math. **7**, 56–74 (1942).

Let $F(t, x, y)$ be continuous in the region $a\leqq t\leqq b$; $-\infty<x<\infty$; $-\infty<y<\infty$, and satisfy there a uniform Lipschitz condition with respect to x and y; that is, two constants K and K' exist such that

$$|F(t, x_1, y_1)-F(t, x_2, y_2)|\leqq K|x_1-x_2|+K'|y_1-y_2|,$$

uniformly in t. The author considers two boundary value problems associated with the differential equation

$$x''+F(t, x, x')=0,$$

namely, (1) $x(a)=A$; $x(b)=B$, where A and B are arbitrary; (2) $x(a)=A$; $x'(b)=\beta$, where A and β are arbitrary. Picard [Leçons sur Quelques Problèmes aux Limites de la Théorie des Équations Différentielles, Gauthier-Villars, Paris, 1930, pp. 2–7] has shown that (1) has one and only one solution if the t-interval (a, b) is so small that $K(b-a)^2/8+K'(b-a)/2<1$. Lettenmeyer proves that this

condition can be improved to $K(b-a)^2/\pi^2+K'4(b-a)/\pi^2<1$ and that this inequality is the "best" in the sense that "$<$" cannot be relaxed to "$=$" if $F(t, x, y)=F(t, x)$ is independent of y, so that K' can be taken to be zero. He also proves that (2) has one and only one solution if the length of the t-interval satisfies $4K(b-a)^2/\pi^2+2K'(b-a)/\pi<1$ and that this is the "best" inequality in the same sense. The proof shows, incidentally, that the inequalities also assure the convergence of the successive approximations. The results are applied to a consideration of the periodic solutions of $x''+k\sin x=l\sin t$, where k and l are constants.

P. Hartman (Baltimore, Md.).

Zwirner, Giuseppe. Ancora sulla teoria delle matrici applicata ai sistemi di equazioni differenziali lineari. Rend. Sem. Mat. Univ. Padova **14**, 37–42 (1943).

Correction of an error in an earlier paper [same Rend. **7**, 55–109 (1936), in particular, p. 101].

Zwirner, Giuseppe. Sull'equazione $y'=\lambda f(x, y)$. Rend. Sem. Mat. Univ. Padova **15**, 33–39 (1946).

The author considers the equation $y'=\lambda f(x, y)$, $y(a)=\alpha$, $y(b)=\beta$, and proves the following theorem using function-space methods. Let $f(x, y)$ be finite and nonnegative in the rectangle R: $a\leqq x\leqq b$, $c\leqq y\leqq d$, measurable with respect to x, and continuous in y. Let there exist, moreover, two nonnegative functions $p(x)$, $q(x)$, such that $p(x)\leqq f(x, y)\leqq q(x)$ in R and $\int_a^b p(x)dx>0$. Then if $c\leqq\alpha$, $\beta\leqq d$, there exists at least one value λ_0 of λ for which the equation possesses at least one absolutely continuous solution in $a\leqq x\leqq b$. Another related result is also proved. *R. Bellman.*

Zwirner, Giuseppe. Criteri d'unicità per gli integrali d'un sistema di equazioni differenziali ordinarie. Rend. Sem. Mat. Univ. Padova **11**, 90–96 (1940).

The author obtains a general functional criterion which insures that a system of first order differential equations

$$(*)\qquad y_i'=f_i(x, y_1, \cdots, y_n),\qquad i=1, \cdots, n,$$

has at most one solution on an interval $\bar{x}\leqq x\leqq b$ satisfying initial conditions $y_i(\bar{x})=\bar{y}_i$, $i=1, \cdots, n$. The theorem extends to systems a result proved by Scorza Dragoni [Rend. Circ. Mat. Palermo **54**, 430–448 (1930)] for a single equation of the first order; moreover, by specialization of the form of the functions involved one obtains the result of Giuliano [Boll. Un. Mat. Ital. (2) **2**, 221–227 (1940); these Rev. **2**, 49] for systems of the form (*). *W. T. Reid.*

Zwirner, Giuseppe. Su un problema di valori al contorno per equazioni differenziali ordinarie di ordine n. Rend. Sem. Mat. Univ. Padova **12**, 114–122 (1941).

This note is concerned with the proof of the existence of a solution of the problem $y^{(n)}=f(x, y, y', \cdots, y^{(n-1)})$, $y(x_1)=c_1, \cdots, y(x_n)=c_n$, for the case in which the function $f(x, y, y', \cdots, y^{(n-1)})$ satisfies certain simple conditions which need not be restated here. The novelty of the work lies in the method employed. The proof is made to depend upon the consideration of transformations in an appropriate function space and upon Brouwer's fixed-point theorem.

L. A. MacColl (New York, N. Y.).

Zwirner, Giuseppe. Un criterio d'esistenza relativo a un problema al contorno per un'equazione differenziale ordinaria d'ordine n. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 217–222 (1942).

The following theorem is stated, with a brief indication of the proof and some discussion of the significance of the theorem. Consider the problem

$$y^{(n)}=f(x, y, y', \cdots, y^{(n-1)}),$$
$$y(x_1)=c_1,\quad y(x_2)=c_2, \cdots,\quad y(x_n)=c_n,$$

where $f(x, \cdots, y^{(n-1)})$ is defined in the region

$$T: a\leqq x\leqq b,\quad |y^{(j)}|<+\infty,\quad j=0, 1, \cdots, n-1;\ y^{(0)}=y,$$

the c's are arbitrarily fixed real numbers, and $a\leqq x_1<\cdots<x_n\leqq b$. If the following conditions (I) and (II) are satisfied, the problem has at least one solution $y=y(x)$, where $y(x)$ and its first $n-1$ derivatives are absolutely continuous in the interval $a\leqq x\leqq b$. (I) $f(x, \cdots, y^{(n-1)})$ is everywhere finite in T, is continuous with respect to $(y, \cdots, y^{(n-1)})$, and is measurable with respect to x. (II) To each positive number M there corresponds a function $\varphi_M(x)$ having the following properties: (1) $\varphi_M(x)$ is summable in $a\leqq x\leqq b$; (2) $\int_a^b\varphi_M(x)dx<1$ for sufficiently large values of M; (3) we have the relation $|f(x, \cdots, y^{(n-1)})|\leqq M\varphi_M(x)$ in the region

$$T_M: a\leqq x\leqq b,\quad |y^{(j)}-G^{(j)}(x)|\leqq\frac{(b-a)^{n-j-1}}{(n-j-1)!}M,$$

where $G(x)$ is a polynomial of degree $n-1$ (at most) satisfying the conditions $G(x_i)=c_i$. *L. A. MacColl.*

Zwirner, Giuseppe. Criteri d'unicità per un problema di valori al contorno per equazioni e sistemi di equazioni differenziali ordinarie d'ordine qualunque. Rend. Sem. Mat. Univ. Padova **13**, 9–25 (1942).

The author states and proves two theorems concerning the uniqueness of the solution of the problem

$$y^{(n)}=f(x, y, y', \cdots, y^{(n-1)}),$$
$$y(x_1)=c_1,\quad y(x_2)=c_2, \cdots,\quad y(x_n)=c_n.$$

He also gives a theorem concerning the uniqueness of the solution of a more general problem of the same kind, involving more than one dependent variable. The statements of the theorems are complicated; they cannot be reproduced conveniently here. *L. A. MacColl* (New York, N. Y.).

Cinquini, S. Problemi di valori al contorno per equazioni differenziali ordinarie. Rend. Sem. Mat. Fis. Milano **14**, 157–170 (1940).

The author reviews various results on the existence of a solution of a differential boundary problem of the form

$$(*)\qquad y^{(n)}=f(x, y, y', \cdots, y^{(n-1)}),\qquad n\geqq 2,$$
$$y(x_i)=y_i,\qquad i=1, \cdots, n,$$

with special attention to work of his own employing the device of approximating a given continuous function f in a system (*) by a function that is Lipschitzian in the arguments $y, y', \cdots, y^{(n)}$. Special theorems which are given for systems (*) with $n=2, 3, 4$ are contained in previous papers of the author [Ann. Scuola Norm. Super. Pisa (2) **8**, 1–22, 271–283 (1939); **9**, 61–77 (1940); Boll. Un. Mat. Ital. (2) **2**, 322–325 (1940); these Rev. **2**, 51, 198]. *W. T. Reid.*

Trevisan, Giorgio. Teoremi di unicità e confronto per problemi relativi a sistemi di due equazioni differenziali ordinarie del primo ordine. Rend. Sem. Mat. Univ. Padova **12**, 12–21 (1941).

The first part of this note is concerned with a system of differential equations of the form $y'=f(x, y, z)$, $z'=g(x, y, z)$, where the functions f and g are real and continuous in S: $a\leqq x\leqq b$, $-\infty<y<+\infty$, $-\infty<z<+\infty$. It is shown that, if f is increasing with respect to z, and g is increasing with respect to y and satisfies a Lipschitz condition with

respect to z, then the system of differential equations has at most one solution satisfying the conditions $z(a)=z_a$, $y(b)=y_b$. It is also shown that, under the same hypotheses, the system of differential equations has at most one solution satisfying the conditions $z(a)=z_a$, $\int_a^b y(x)dx=K$. Finally, there is given a rather complicated theorem concerning the comparison of solutions of two systems of equations of the above form. *L. A. MacColl* (New York, N. Y.).

Scorza Dragoni, Giuseppe. Teoremi di unicità relativi a un problema al contorno per un sistema di due equazioni differenziali, ordinarie, del primo ordine. Rend. Sem. Mat. Univ. Padova **12**, 30–50 (1941).

The author proves various theorems, relating to the existence and uniqueness of solutions of systems of differential equations, which are extensions and generalizations of the results given by Trevisan in the paper reviewed above. Thus, whereas Trevisan required two functions $f(x, y, z)$, $g(x, y, z)$ to be continuous with respect to (x, y, z), the present author merely requires the functions to be continuous with respect to (y, z) and measurable with respect to x. Much of the attention is given to the existence theorems, which were not considered in Trevisan's note.

L. A. MacColl (New York, N. Y.).

Scorza Dragoni, Giuseppe. A proposito di alcuni teoremi sulle equazioni differenziali ordinarie. Rend. Sem. Mat. Univ. Padova **15**, 60–131 (1946).

The author replies to criticisms of his work made by Cinquini.

★**Minorsky, N. Introduction to Non-Linear Mechanics. I. Topological Methods of Non-Linear Mechanics.** David Taylor Model Basin. Rep. no. 534. Washington, D. C., 1944. 133 pp.

★**Minorsky, N. Introduction to Non-Linear Mechanics. II. Analytical Methods of Non-Linear Mechanics.** David Taylor Model Basin. Rep. no. 546. Washington, D. C., 1945. 113 pp.

★**Minorsky, N. Introduction to Non-Linear Mechanics. III. Non-Linear Resonance.** David Taylor Model Basin. Rep. no. 558. Washington, D. C., 1946. 132 pp.

Applications of the mathematical work of Poincaré, Liapounoff, Birkhoff, Kryloff and Bogoliuboff, and others are given following van der Pol and a group of Russians including Mandelstam, Papelexi and Andronow. "This report [including Part IV which has not as yet appeared] reviews the progress accomplished in Non-Linear Mechanics approximately up to 1940; its preparation was greatly facilitated by the availability of the following two works in the Russian language. 1. 'Theory of Oscillations,' by A. Andronow and S. Chaikin, Moscow, 1937. 2. 'Introduction to Non-Linear Mechanics,' by N. Kryloff and N. Bogoliuboff, Kief, 1937."

Part I is on applications of the Poincaré-Bendixon topological results and is a condensation of "Theory of Oscillations" with the results of O. K. Smith and the reviewer on the Liénard equation added. The entire report is written for the engineer and physicist rather than the mathematician. Thus many proofs are omitted. An example, usually from electricity but sometimes from mechanics, is given to illustrate the origin of each differential equation studied.

In part II the procedural details of the methods of small parameters of Poincaré, Linstedt, van der Pol, and Kryloff and Bogoliuboff are given. The equations treated are mainly of the form $\ddot{x}+x=\mu f(x, \dot{x})$ with μ small. [The remark on page 76 concerning divergence in the case of the method of Gylden and Linstedt is misleading since, for the application made there, the method gives a convergent result.]

"Part III deals with the complicated phenomena of nonlinear resonance with its numerous ramifications such as internal and external subharmonic resonance, entrainment of frequency, parametric excitation, etc. . . . Much credit for the experimental and theoretical studies of these phenomena is due to Mandelstam and Papelexi, and the school of physicists under their leadership. The first four chapters of Part III represent the applications of the quasi-linear theory of Kryloff and Bogoliuboff to these problems and the last three concern the developments of Mandelstam, Papelexi, Andronow, Witt, and others, following the classical theory of Poincaré." *N. Levinson.*

Teodorchik, K. On the theory of sinusoidal auto-oscillations in systems with many degrees of freedom. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 33–36 (1946).

The author uses the Nyquist diagram method to study the quasi-linear differential equations of a self-excited amplifier. *R. Bellman* (Princeton, N. J.).

Teodorchik, K. Limits of applicability of van-der-Pohl's method. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 123–126 (1946).

The author investigates various equations of the form

$$\ddot{x}-\phi(x, \dot{x}, \ddot{x}, \cdots)+\omega_0^2 x=0$$

and determines conditions under which van der Pol's method can be used to find approximate harmonic solutions of the form $x=a \sin \omega t$, where a and ω are slowly varying functions of the time. *R. Bellman* (Princeton, N. J.).

Bulgakov, B. V. Problems of the control theory with nonlinear characteristics. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 313–332 (1946). (Russian. English summary)

The author considers systems

$$\sum_{k=1}^{n} f_{jk}(D)x_k=\psi_j(x_1, \cdots, x_n, t), \quad j=1, \cdots, n,$$

where $D=d/dt$ and the f_{jk} are polynomials with constant coefficients. General assumptions are made, among them that the linear approximation has distinct roots (not necessarily complex). The behavior of the nonlinear terms is of the same type as of those studied by Andronoff and Chaikin and by Bogoliuboff and Kryloff, and the approximate quasi-harmonic solutions are dealt with using methods inspired by these authors. Application is made to the regulation problem with two second-order equations in two unknowns. *S. Lefschetz* (Princeton, N. J.).

Taeger, W. Die Entdämpfung von Schwingungskreisen durch Eisendrosseln. Arch. Elektrotechnik **35**, 193–216 (1941).

If the self-inductance in an electric circuit depends on the intensity of the current (for instance, in consequence of an iron core in a coil), the differential equation of the oscillations is nonlinear. The author puts $\varphi=\int L di$ (L self-inductance, i intensity of current) and discusses the differential equation for the case when i is an odd cubic polynomial in φ; there are also indications of the extension of the method to the case when i is an odd polynomial of any degree in φ. In either case the differential equation is "pseudo-harmonic"

(nonlinear with constant coefficients) and can be approximated by a "quasiharmonic" equation, that is, a linear equation with periodic coefficients, and this latter equation can be transformed to Hill's form. There follows a summary of the (mostly well-known) properties of the Hill equation, including the formula for the solution of the nonhomogeneous equation ("forced oscillations"). The most interesting results of the paper are on self-excitation of such a system. The conditions under which self-excited oscillations originate are investigated, and the calculations are carried through far enough to make numerical applications possible.

A. Erdélyi (Edinburgh).

Calamai, Giulio. Sulla stabilità delle soluzioni per l'equazione differenziale del secondo ordine a coefficienti periodici. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 183–193 (1942).

The stability of the solutions of the linear differential equation $\ddot{x}(t)+p(t)\dot{x}(t)+q(t)x(t)=0$ with periodic coefficients, of period ω, is considered by an analysis of the successive approximations obtained in using the Picard method. It is shown that a sufficient condition for stability is given by $p(t)\geqq 0$, $p(0)=0$; $q(t)\geqq 0$ in $(0,\omega)$ without vanishing identically in any interval; and, for t in $(0,\omega)$,

$$t\int_0^t q(\tau)d\tau<1, \qquad p(t)\int_0^t q(\tau)d\tau<q(t)\int_0^t \tau q(\tau)d\tau,$$

$$\int_0^t p(\tau)d\tau<1-\int_0^t \tau q(\tau)d\tau, \qquad p(t)<q(t)\int_0^t \tau^2 q(\tau)d\tau.$$

N. Levinson (Cambridge, Mass.).

Yelchin, M. Sur les conditions pour qu'une solution d'un système linéaire du second ordre possède deux zéros. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 573–576 (1946).

The paper is concerned with the differential equation $Y''=I(x)Y$, on an interval (a,b) upon which $I(x)$ is continuous. It gives a necessary and sufficient condition that the solution determined by values $Y(x_0)=1$, $Y'(x_0)=\psi(x_0)$, where x_0 is an inner point of (a,b), have at least two zeros in this interval. Upon the result is based a criterion for the oscillatory character of the solutions when $I(x)$ is periodic.

R. E. Langer (Madison, Wis.).

Armellini, Giuseppe. Sopra l'equazione differenziale del moto centrale newtoniano. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 342–348 (1943).

In the equation $d^2r/dt^2=c^2r^{-3}-M(t)r^{-2}$, $M(t)$ is positive and nondecreasing for $t\geqq 0$. At $t=0$, $r(0)>0$ and $\dot{r}(0)$ is real. As $t\to\infty$, $M(t)\to\infty$ at least as fast as the first power of t. Moreover, $\log M(t)$ tends "regularly" to infinity with t. Then the author shows that $r(t)\to 0$ as $t\to\infty$. The "regular" growth of $\log M(t)$ means that, even if the growth in a set of intervals of density less than ϵ in $(0,\infty)$ is excluded, $\log M(t)\to\infty$.

N. Levinson (Cambridge, Mass.).

Schwerdtfeger, H. The eigen-value problem of Hill's equation. J. Proc. Roy. Soc. New South Wales **79**, 176–189 (1946).

After some introductory remarks on eigenvalues in quantum and classical mechanics, Hill's differential equation with a bounded and continuous coefficient function is considered under the boundary condition that the solution is bounded for all values of the independent variable. The investigation is based on the work of H. A. Kramers [Physica **2**, 483–490 (1935)] and makes use of the matrix notation for sets of independent solutions. The main properties proved by Kramers are reviewed in this notation. Special attention is given to periodic solutions and to asymptotic representations of the corresponding eigenfunctions.

M. J. O. Strutt (Eindhoven).

Cesari, Lamberto. Sulla stabilità delle soluzioni dei sistemi di equazioni differenziali lineari a coefficienti periodici. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **11**, 633–695 (1941).

The author considers a system of linear homogeneous differential equations

$$y_i'=\sum_{k=1}^{n}\{a_{ik}+\lambda\varphi_{ik}(x)\}y_k(x), \qquad i=1,\cdots,n,$$

in which $\varphi_{ik}(x)$ are real periodic functions of period $T=2\pi/\omega$ and of average value zero, while λ is a real parameter. Particular cases are the equations of Mathieu and of Hill. The investigation aims at the statement of necessary and sufficient conditions, bearing on the coefficients of the differential system, leading to a system of stable solutions, i.e., solutions bounded for all positive values of the independent variable. Introducing characteristic exponents of a system of independent integrals, such stability exists if the real parts of all n exponents are zero or negative, while they are different modulo ωi. This is verified in the simple case $i=1$. Using matrix symbolism, the Liouville series solving this simple case is extended to the general case and utilized for obtaining upper bounds for the characteristic exponents. By a modification of this algorithm a direct evaluation of characteristic exponents is effected in the general case and by comparison with a given simple system of differential equations closer bounds for these exponents are obtained. These results are then applied to differential systems of the second order and particularly to Hill's equation, where specific theorems on stability or instability of solutions as dependent on parameter values are derived. Systems of higher order are then considered with a view to the derivation of similar theorems. A few examples are given in order to show that the limitations imposed in the theorems are essential.

M. J. O. Strutt (Eindhoven).

Makai, E. Eine Eigenwertabschätzung bei gewissen Differentialgleichungen zweiter Ordnung. Mat. Fiz. Lapok **48**, 510–532 (1941). (Hungarian. German summary)

Let $\rho(x)$ be positive and have a second derivative continuous in stretches in $a\leqq x\leqq b$. We denote by λ_n and λ_n' the nth characteristic values of the differential equation $y''+\lambda\rho(x)y=0$ with the boundary conditions $y(a)=y(b)=0$ and $y'(a)=y'(b)=0$, respectively. According to a classical result of Liouville,

$$\lambda_n\sim\lambda_n'\sim\left\{\pi n\Big/\int_a^b\{\rho(x)\}^{\frac{1}{2}}dx\right\}^2=\Lambda_n, \qquad n\to\infty.$$

The author assumes, in addition to the conditions mentioned, that $\rho\rho''-\rho'^2<0$ (this condition is satisfied if ρ is convex from above). Then $\lambda_n<\Lambda_n$. Similarly if $\rho\rho''-\rho'^2>0$ we have $\lambda_n'<\Lambda_n$. Under the same conditions the following inequalities hold:

$$\lambda_n'<((n+1)/n)^2\Lambda_n, \quad \lambda_n<((n+1)/n)^2\Lambda_n,$$

respectively. As an application, bounds for the zeros of Hermite polynomials are obtained.

G. Szegö.

Ważewski, Tadeusz. Théorie des multiplicités régulières d'éléments de contact unis. Application aux transformations canoniques. Ann. Soc. Polon. Math. **18**, 55–112 (1945).

If the set of functions

$$(1)\qquad x^i=x^i(u^1,\cdots,u^m),\quad y^i=y^i(u^1,\cdots,u^m),\quad s=s(u^1,\cdots,u^m),$$

$i=1,\cdots,n$, are of class C^1, the rank of the Jacobian matrix

$$\|D(x^1,\cdots,x^n,s,y^1,\cdots,y^n)/D(u^1,\cdots,u^m)\|$$

is m, and if they satisfy (2) $ds=\sum_{i=1}^n y^i dx^i$, then (1) is called a regular solution of (2). The paper gives a method of constructing the most general regular solution of (2). Previous methods of finding solutions of (2) either did not give the general solution or required the x^i of (1) to be of class C^2. The author applies the results of the first part of the paper to obtain all canonical transformations of class C^1 [see C. Carathéodory, Variationsrechnung, Teubner, Leipzig-Berlin, 1935, pp. 78–102, for another method of obtaining canonical transformations of class C^1]. *F. G. Dressel.*

Coutrez, Raymond. Sur les variétés caractéristiques des équations aux dérivées partielles du second ordre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **28**, 266–282 (1942). [MF 13662]

The author discusses the local solution of the Cauchy problem associated with a second order partial differential equation in n variables, with initial values prescribed on an $(n-1)$-dimensional hypersurface V_{n-1}. Obtaining particularly transparent forms for the characteristic conditions, he analyzes the problem that results when V_{n-1} is characteristic: it is shown that in this case a solution is uniquely determined when, in addition to the Cauchy data, the value of one derivative of each order is prescribed on V_{n-1}, each such value being subject, of course, to a condition of consistency. *J. W. Calkin* (Houston, Tex.).

Coutrez, Raymond. Sur les variétés caractéristiques d'un système d'équations aux dérivées partielles d'ordre quelconque. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 15–30 (1943). [MF 13836]

The methods of the paper reviewed above are adapted to the general case of a system of partial differential equations of arbitrary order and analogous results are obtained. Again there is special reference to the nature of supplementary conditions which may be used to specify a unique solution when Cauchy data are given on a characteristic hypersurface. *J. W. Calkin* (Houston, Tex.).

Coutrez, Raymond. Sur les variétés caractéristiques d'un système d'équations aux dérivées partielles d'ordre quelconque. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 71–78 (1943). [MF 13839]

Where two distinct solutions of the system of part I [cf. the preceding review] are considered on the respective sides of the surface V_{n-1}, one has in effect a solution of the system with discontinuities on V_{n-1}. If the order of the system is p, the discontinuities in the derivatives of order not less than p are recursively functions of the discontinuities in the derivatives of lower order, and of one of the two sets of Cauchy data on V_{n-1}. If V_{n-1} is characteristic for the system, an analogous situation obtains with reference to the boundary problem with supplemented Cauchy data treated in part I; this is also discussed. *J. W. Calkin.*

Grünberg, G. A new method of solution of certain boundary problems for equations of mathematical physics permitting of a separation of variables. Acad. Sci. USSR. J. Phys. **10**, 301–320 (1946).

Translation of a paper in Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **10**, 141–168 (1946); these Rev. **7**, 521]. *R. P. Boas, Jr.* (Providence, R. I.).

Cinquini-Cibrario, Maria. Una proprietà degli integrali delle equazioni ellittico-paraboliche del secondo tipo misto. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 502–510 (1942).

The differential equation considered is

$$(A)\qquad (x^{2(k+1)}/k^2)z_{xx}+z_{yy}=0$$

for a positive integer k. Equation (A) is elliptic except for the points of the y-axis. Hence solutions of (A) defined in a region S are necessarily analytic at interior points of S which do not lie on the y-axis. In general, solutions of (A) will be represented by different analytic functions in the two half-planes $x>0$, $x<0$. Continuation of solutions across the y-axis is in general not unique.

The main result is that a solution of (A) which is regular analytic in a single point of the y-axis is necessarily analytic in all points of the y-axis. This is proved by showing that a solution $z(x, y)$ of (A) is analytic in the parallel strip $|x|<R$, if $z(x, 0)$ and $z_y(x, 0)$ are given by convergent power series for $|x|<R$. For this purpose $z(x, y)$ is expanded in terms of special solutions $U_{n,k}$ and $V_{n,k}$, for which the Cauchy data on the x-axis reduce to a single power x^n. A discussion of the $U_{n,k}$ and $V_{n,k}$ shows that the expansion for z converges and represents an analytic function in the whole strip $|x|<R$. *F. John* (New York, N. Y.).

Krzyżański, Mirosław. Sur les solutions de l'équation linéaire du type parabolique déterminées par les conditions initiales. Ann. Soc. Polon. Math. **18**, 145–156 (1945).

A function $g(x_1,\cdots,x_n)=g(x)$ will be said to belong to the class E if it is continuous and there exist two positive constants M and K such that $|g(x)|\leqq M\exp(K\sum_{i=1}^n x_i^2)$. Let R be the infinite region $0\leqq y\leqq h$, $-\infty<x_i<\infty$, $i=1,\cdots,n$, and let $F(u)$ be the parabolic operator

$$\sum_{i,j=1}^n a_{ij}(x,y)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^n a_i(x,y)\frac{\partial u}{\partial x_i}+a(x,y)u-\frac{\partial u}{\partial y},$$

where $\sum_{i,j=1}^n a_{ij}\lambda_i\lambda_j$ is a positive definite form. Two earlier results of the author on the uniqueness of solutions of the equation $F(u)=0$ are repeated [Bull. Amer. Math. Soc. **47**, 911–915 (1941); these Rev. **3**, 246; the criticism given in the review does not apply to the present paper as the continuity conditions on $a_i(x, y)$ and $a(x, y)$ have been strengthened]. The paper also proves that, if a_{ij}, a_i, a are bounded and satisfy a Lipschitz condition in R, then the equation $F(u)=f(x, y)$ has a solution belonging to the class E which vanishes for $y=0$ if $f(x, y)$ belongs to the class E. [In the statements, p. 156, on maximum and minimum values the author must refer only to solutions of class E.]

F. G. Dressel (Durham, N. C.).

Paterson, Stewart. On certain integrals in the theory of heat conduction. Quart. Appl. Math. **4**, 305–306 (1946).

As a remark on the note by W. Horenstein under the same title [same Quart. **3**, 183–184 (1945); these Rev. **6**, 225] the author calls attention to the well-known fact that

the use of the Laplace transform in solving certain types of problems in heat conduction leads directly to results in terms of error functions and other tabulated functions.

R. V. Churchill (Ann Arbor, Mich.).

Zwirner, Giuseppe. Teoremi di confronto per l'equazione differenziale $\partial^2 z/\partial x\partial y = f(x, y, z)$. Atti Mem. Accad. Sci. Padova. Mem. Cl. Sci. Fis.-Mat. (N.S.) **59**, 45–55 (1943).

A theorem of Bompiani [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **1**, 298–302 (1925)] on ordinary differential equations is extended to prove the following result for the differential equation (*) $z_{xy} = f(x, y, z)$. Suppose that for (x, y) on R: $a \leqq x \leqq b$, $c \leqq y \leqq d$ and $|z| \leqq N$ the function $f(x, y, z)$ is defined and satisfies for every pair of values z_1, z_2 with $-N \leqq z_1 \leqq z_2 \leqq N$ the inequality $f(x, y, z_2) - f(x, y, z_1) \leqq \phi(x, y, z_2 - z_1)$, where $\phi(x, y, u)$, $u \geqq 0$, is a continuous function of (x, y, u) that is non-decreasing in u, and for which there is an $M > 0$ such that $|\phi(x, y, u)| < M$ for (x, y) on R, $0 \leqq u \leqq (b-a)(d-c)M$. If $z = z_1(x, y)$ and $z = z_2(x, y)$ are solutions of (*) which coincide along $x = a$, $c \leqq y \leqq d$ and $y = c$, $a \leqq x \leqq b$, while $z_1(x, y) \leqq z_2(x, y)$ on R, then $0 \leqq z_2(x, y) - z_1(x, y) \leqq u_0(x, y)$, where $u_0(x, y)$ is the maximal solution on R of $u_{xy} = \phi(x, y, u)$, $u_0(a, y) = 0 = u_0(x, c)$. This result is extended to give a corresponding bound for the difference of two solutions $z = \tau_1(x, y)$, $z = \tau_2(x, y)$ of (*) which coincide along characteristic segments $x = x_0$, $c \leqq y \leqq d$, and $y = y_0$, $a \leqq x \leqq b$, while $\tau_1(x, y) \leqq \tau_2(x, y)$ on R. *W. T. Reid* (Evanston, Ill.).

Petrashen, G. The establishment of oscillatory wave regimes and the resonance phenomenon in the case of a spherical domain. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 13–16 (1946).

The solution of the initial value problem for the vector wave equation for a spherical domain leads, as the author has shown [same C. R. **46**, 266–269 (1945); these Rev. **7**, 16] to a Volterra equation of the second kind with a kernel depending on a difference. In the present paper, the equation is solved by the use of the method of Laplace transforms for a disturbance $a(t)$ specified at radius $r = 1$ and starting at time $t = 0$. The solution is given in closed form when $a(t)$ is a polynomial. *H. Feshbach.*

Petrashen, G. Wave processes in a spherical domain in the case of normal absorption. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 187–190 (1946).

The procedures developed in preceding papers [cf. same C. R. **46**, 266–269 (1945); these Rev. **7**, 16] are now applied to the equation

$$\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{n(n+1)}{r^2} f = \frac{\partial^2 f}{\partial t^2} + 2\lambda \frac{\partial f}{\partial t}$$

which arises for wave propagation in a spherical domain with absorption. The solutions for arbitrary initial conditions are exhibited in closed form. The solutions are discussed for small λ. *H. Feshbach* (Cambridge, Mass.).

Petrashen, G. The application of the method of spherical vectors to problems of diffraction of elastic disturbances. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 587–590 (1946).

The methods developed earlier [same C. R. **46**, 266–269 (1945); these Rev. **7**, 16] are now applied to the vector wave equation describing disturbances in an elastic medium.

H. Feshbach (Cambridge, Mass.).

Fantappiè, Luigi. Risoluzione in termini finiti del problema di Cauchy, con dati iniziali su una ipersuperficie qualunque. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 948–956 (1941).

In Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **8**, 613–653 (1937) the author has given the solution of the general linear equations of order m in $n+1$ independent variables $x_0, \cdots, x_n$, with constant coefficients, which may be written in the form

$$\frac{\partial^m z}{\partial x_0^m} + \sum a_{r_0} a_{r_1} \cdots a_{r_n} \frac{\partial^{r_0 + r_1 + \cdots + r_n} z}{\partial x_0^{r_0} \partial x_1^{r_1} \cdots \partial x_n^{r_n}} = f(x_0, \cdots, x_n),$$

when the values of $z, \partial z/\partial x_0, \cdots, \partial z^{m-1}/\partial x_0^{m-1}$ are given for $x_0 = 0$ and, more generally, when the analogous data are given on an arbitrary linear $[n]$-space. Here he deals with the same problem, when instead of the linear $[n]$-space occurs an arbitrary (noncharacteristic) $[n]$-space, given by $x_0 = \psi(x_1, \cdots, x_n)$. The solution is obtained by a finite number of integrations and written in a concise symbolic form by introducing the operators I and B_s given by

$$If = \int_\psi^{x_0} f(t, x_1, \cdots, x_n)\, dt, \quad B_s f = I \frac{\partial}{\partial x_s} f.$$

H. Bremekamp (Delft).

Difference Equations, Special Functional Equations

de Amorim, Pacheco. Symbolic calculus. Revista Fac. Ci. Univ. Coimbra **8**, 50–88 (1940). (Portuguese)

Continuation of an exposition in the same Revista **5**, 229–251, 373–404 (1935); **6**, 87–118 (1936), 200–230 (1937), 466–479 (1938). This part is mainly concerned with the calculus of finite differences.

Andreoli, Giulio. Sull'equazione di Stiefel ed altre analoghe. Rend. Accad. Sci. Fis. Mat. Napoli (4) **12**, 48–53 (1942).

Stiefel's relation which expresses a binomial coefficient as the sum of two others of lower base is considered as a difference equation. The principal properties of the solution of this and other analogous equations are described.

From the author's summary.

Cowan, R. W. Solution of a linear difference equation of the second order with polynomial coefficients of degree n. Revista Ci., Lima **48**, 19–31 (1946).

The author solves

$$A_1(x) f(x+2) + A_2(x) f(x+1) + A_3(x) f(x) = 0,$$

where $A_i(x) = \sum_{k=0}^{n} a_{ik} x^{n-k}$, $i = 1, 2, 3$, for the special case where the characteristic equation $a_{10}\rho^2 + a_{20}\rho + a_{30} = 0$ has distinct roots r_1, r_2 with $|r_1/r_2| < 1$ and r_1 satisfies $a_{11}\rho^2 + a_{21}\rho + a_{31} = 0$. By use of symbolic methods and utilization of the infinite integral formula for the generalized Riemann zeta function, a solution is obtained in the form $f(x) = \alpha_0 r_1^x + \sum_{n=1}^{\infty} \alpha_n(x)$, where α_0 is an arbitrary constant and the functions $\alpha_n(x)$ are solutions of nonhomogeneous difference equations with constant coefficients; $\alpha_n(x)$ is expressed as a finite sum of n-fold infinite integrals. By virtue of the conditions imposed on r_1 and r_2, the series for $f(x)$ is shown to be convergent for $|x|$ sufficiently large.

D. Moskovitz (Pittsburgh, Pa.).

Mambriani, Antonio. Equazioni lineari e omogenee alle differenze finite aventi soluzioni polinomiali. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 563–571 (1942).

The author considers a difference equation of the form

$$(A)\quad p_{m-1}\Delta^m y+p_{m-2}\Delta^{m-1}y+\cdots+p_1\Delta^2 y+(a_1x+b_1)\Delta y+(a_0x+b_0)y=0,$$

where $\Delta y=y(x+1)-y(x)$ and p_k denotes a polynomial of degree k, $k=1,\cdots,m-1$. Here a_1, a_0, b_1, b_0 are constants with a_0, b_0 not both vanishing. The author proves that a necessary and sufficient condition for the existence of a polynomial of degree n satisfying equation (A) is that $a_0=0$, $b_0=-na_1$. The proof makes use of expansions into factorials, in terms of which the Δ-operator can be inverted explicitly.

F. John (New York, N. Y.).

Ghermănescu, M. A class of functional equations. Pozitiva **1**, 121–125 (1940). (Romanian. French summary)

L'équation fonctionnelle

$$E[\varphi]=\sum_1^n A_i\varphi(x_i, x_{i+1}, \cdots, x_n, x_1, x_2, \cdots, x_{i-1})=0$$

où au moins deux des constantes A_i sont différentes de zéro, admet des solutions non nulles si, et seulement si, parmi les racines de l'équation caractéristique

$$E(x)=A_1+A_2x+\cdots+A_nx^{n-1}=0$$

il y a au moins une racine de l'équation $1-x^n=0$.

Author's summary.

Popovici, Constantin. Problèmes proposés concernant certaines analogies. Mathematica, Timişoara **22**, 147–154 (1946).

The author quotes various older results to show the close analogy between functional equations and differential equations. In addition, he discusses the second order functional equation for f, (1) $f+af_1+bf_2=q$, where $f_1(x)=f(x_1)$, $f_2(x)=f_1(x_1)$ and $x_1=x_1(x)$ is a transformation whose third iterate is the identity. In particular, the author determines all coefficients a, b for which the solution of (1) becomes nontrivial, that is, for which the determinant

$$\begin{vmatrix}1 & a & b\\ b_1 & 1 & a_1\\ a_2 & b_2 & 1\end{vmatrix}$$

vanishes.

F. John (New York, N. Y.).

Beke, E. Über eine Funktional-Differentialgleichung. Mat. Fiz. Lapok **48**, 387–392 (1941). (Hungarian. German summary)

Die Formel $\sin 2x=2\sin x\cos x=2\sin x(\sin x)'$ legt die Frage nahe, welche die analytischen Funktionen sind, die sich in $x=0$ regulär verhalten und die Funktional-Differentialgleichung $f(2x)=2f(x)f'(x)$ befriedigen. Das Ergebnis ist, dass nur 0, x, $c^{-1}\sin cx$, und $ce^{x/2c}$ die gesuchten Funktionen sind. Das Problem wird in speziellen Fällen auch für $f(cx)=cf(x)f'(x)$ gelöst.

Author's summary.

Integral Equations

Zaanen, A. C. On the theory of linear integral equations. IV. Nederl. Akad. Wetensch., Proc. **49**, 409–416 = Indagationes Math. **8**, 264–271 (1946). [MF 16823]

[The first three parts appeared in the same Proc. **49**, 194–204, 205–212, 292–301 = Indagationes Math. **8**, 91–101, 102–109, 161–170 (1946); these Rev. **8**, 28.] The author applies the results of parts I and II to integral equations of the second kind with an L^2 kernel $K(x, y)$, which is symmetrisable in the sense that there exists a nonnegative definite Hermitian L^2 kernel $H(x, y)$ such that the kernel $P(x, y)=\int H(x, z)K(z, y)dz$ is Hermitian. If also every L^2 function $f(x)$ such that $\int H(x, y)f(y)dy=0$ also satisfies $\int K(x, y)f(y)dy=0$, K is called a Marty kernel. The author shows how the classical theory of integral equations with symmetrisable kernels may be generalised to fit kernels of this type. In particular, the expansion theorem for the kernel assumes the form

$$\lim_{k\to\infty}\int\int\int H(z, y)\left\{\overline{K(x, z)}-\sum_{i=1}^{k}\lambda_i\overline{\psi_i(x)}\chi_i(z)\right\}\times\left\{K(y, z)-\sum_{i=1}^{k}\lambda_i\psi_i(y)\overline{\chi_i(z)}\right\}dxdydz=0,$$

where $\{\lambda_i\}$ is the set of eigenvalues and $\{\psi_i\}$ the set of eigenfunctions of K and $\chi_i(x)=\int H(x, y)\psi_i(y)dy$.

F. Smithies (Cambridge, England).

Zaanen, A. C. On the theory of linear integral equations. IVa. Nederl. Akad. Wetensch., Proc. **49**, 417–423 = Indagationes Math. **8**, 272–278 (1946). [MF 16824]

This is an immediate continuation of part IV [see the preceding review]. The author shows that if, for some n, the series $\sum_i\lambda_i^n\psi_i(x)\overline{\chi_i(y)}$ is convergent in mean square to $S_n(x, y)$, then $S_n(x, y)=K_n(x, y)-p_n(x, y)$, where K_n is the nth iterate of K and $\int H(x, z)p_n(z, y)dz=0$. Furthermore, if $p\geqq 1$, the series $\sum_i\lambda_i^{n+p}\psi_i(x)\overline{\chi_i(y)}$ is convergent in mean square to $K_{n+p}(x, y)$. It follows, in particular, that $\int K_{n+p}(x, x)dx=\sum_i\lambda_i^{n+p}$ for $p\geqq 2$. If the operator K is of the form $K=AH$, where A is bounded and self-adjoint, and H has a Hermitian L^2 kernel, n can be taken equal to 2; if, in addition, $H=G^2$, where G has a Hermitian L^2 kernel, n can be taken equal to 1.

F. Smithies.

Trjitzinsky, W. J. Singular integral equations with Cauchy kernels. Trans. Amer. Math. Soc. **60**, 167–214 (1946).

The paper deals with equations of the type

$$(\pi i)^{-1}\int_L(y-t)^{-1}K(t, y)\phi(y)dy=f(t),$$

$$\alpha(t)\phi(t)+(\pi i)^{-1}\int_L(y-t)^{-1}K(t, y)\phi(y)dy=f(t),$$

in which $K(t, y)$, $\alpha(t)$, $f(t)$ are given and L consists of a finite number of directed open arcs and closed curves with only isolated points in common and continuously turning tangents except at a finite number of "z-points" at which L may have two distinct tangents or several arcs or closed curves intersect. The author proves that under quite general conditions both equations can be reduced to regular Fredholm equations of the second type. This extends the work done on the same problem by Michlin [C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 315–317 (1939); these Rev. **2**, 100], N. I. Mushelišvili [Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **10**, 1–43 (1941); these Rev. **4**, 160], I. Vecoua [ibid., 45–72 (1941); these Rev. **4**, 160], F. Gahov [Hachoff] [Rec. Math. [Mat. Sbornik] N.S. **2(44)**, 673–683 (1937)] and others, the essentially new contribution being the admittance of z-points, which were excluded in previous work. In order to deal with z-points, the author extends the notion of a Cauchy principal value of the inte-

gral $\Phi(x)=\int_L(y-x)^{-1}F(y)dy$ to cases where x is a z-point and investigates the relation between these principal values and the value of $\Phi(x)$ for neighboring values of x in the open complement of L. In particular, he generalizes to the case when t is a z-point the well-known Plemelj formulae $\Phi^+(t)=\Phi(t)+\pi iF(t)$, $\Phi^-(t)=\Phi(t)-\pi iF(t)$, where $\Phi^+(t)$ and $\Phi^-(t)$ are the limits of $\Phi(t+we)$ as $t+we\to t$ from the positive and negative sides of L, respectively. *H. R. Pitt.*

Consiglio, A. Risoluzione di una equazione integrale non lineare presentatasi in un problema di turbolenza. Atti Accad. Gioenia Catania (6) **4**, no. XX, 13 pp. (1940).

The equation to be solved for $\beta(x)$ is

$$1-\int_0^x\beta(y)(x-y)^{-\frac{1}{2}}dy=B(\pi\lambda)^{\frac{1}{2}}(\beta(x))^{\frac{1}{2}},$$

which by substitutions is reduced to

$$1-\int_0^s\alpha^2(t)(s-t)^{-\frac{1}{2}}dt=\alpha(s).$$

By employing the method of successive substitution, starting with the first approximation $\alpha_0(s)=1$, it is demonstrated that there exists a solution of this equation, proceeding in powers of $s^{\frac{1}{2}}$, convergent for $|s|<\frac{1}{4}$. *T. H. Hildebrandt.*

Eriksson, H. Adolf S. On the distribution of the neutron energies in a moderator of infinite size. Ark. Mat. Astr. Fys. **33B**, no. 5, 8 pp. (1946).

This paper is concerned with the determination of the function $F(E)$ defined by the equations

$$(1)\qquad F(E)=\frac{1}{1-\alpha^2}\int_E^{E/\alpha^2}E'^{-1}F(E')dE',\qquad E\leqq\alpha^2E_0,$$

$$F(E)=\frac{1}{1-\alpha^2}\left\{Q/E_0+\int_E^{E_0}E'^{-1}F(E')dE'\right\},\qquad \alpha^2E_0\leqq E\leqq E_0.$$

These equations are equivalent to equations (8a, b) of a recent paper by Placzek [Phys. Rev. (2) **69**, 423–438 (1946); these Rev. **8**, 30]. Eriksson's method of solution is rather different from that of Placzek.

If one writes $E=E_0e^{-s}$, $F(E)=f(x)$, $a=\log(1/\alpha^2)$, $q=Q/E_0$, equations (1) become

$$(2a)\qquad f(x)=\frac{1}{1-\alpha^2}\int_{x-a}^x f(x')dx',\qquad x\geqq a,$$

$$(2b)\qquad f(x)=\frac{1}{1-\alpha^2}\left\{q+\int_0^x f(x')dx'\right\},\qquad 0\leqq x\leqq a.$$

The solution of (2b) is $f_0(x)=q(1-\alpha^2)^{-1}e^{x/(1-\alpha^2)}$. The solution of (2a) for the range $a\leqq x\leqq 2a$ is obtained by solving the equivalent differential equation

$$(3)\qquad f_1'(x)-(1-\alpha^2)^{-1}f_1(x)=-(1-\alpha^2)^{-1}f_0(x)e^{-a/(1-\alpha^2)}.$$

The arbitrary constant in the solution of (3) is obtained by substitution in (2a) and carrying out the integration. The same procedure is next applied to the range $2a\leqq x\leqq 3a$, and so on. *E. T. Copson* (Dundee).

Scorza Dragoni, Giuseppe. Un'osservazione su un problema al contorno per un'equazione integro-differenziale. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **101**, 695–710 (1942).

The observation concerns the boundary value problem of the nth order integrodifferential equation

$$y^{(n)}(x)=F\left\{x, y(x), \cdots, y^{(n-1)}(x); \int_a^x g(x, v, y(v), y'(v), \cdots, y^{(n-1)}(v))dv\right\}$$

with $y(x_i)=c_i$, $i=0, 1, \cdots, n-1$, a generalization of the second order case considered by L. Baeri [Rend. Circ. Mat. Palermo **44**, 103–138 (1920)] and S. Cinquini [Ann. Mat. Pura Appl. (4) **20**, 257–270 (1941); these Rev. **7**, 452]. It is pointed out that the existence of a solution of this problem can be reduced to the question of a fixed point of a functional transformation $\varphi(x)=T(\varphi(x))$ with a suitably defined T to which the Birkhoff-Kellogg methods as used by Schauder and Cacciopoli are applicable.

T. H. Hildebrandt (Ann Arbor, Mich.).

Zwirner, Giuseppe. Sopra i problemi di valori al contorno per una particolare classe di equazioni integro-differenziali. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **102**, 351–368 (1943).

The author points out a method for reducing the existence theorem for an integro-partial-differential equation

$$D\frac{\partial^{p+q-2}z(x, y)}{\partial x^{p-1}\partial y^{q-1}}=F\left\{x, y, z(x, y), \frac{\partial^{p+q}z}{\partial x^p\partial y^q}; \int_a^x\int_c^y g\left(x, y, z(u, v), \frac{\partial^{p+q}z}{\partial u^p\partial v^q}\right)dudv\right\}$$

with boundary conditions $z(x_i, y)=\varphi_i(y)$; $z(x, y_i)=\psi_i(y)$; $a\leqq x_i\leqq b$, $c\leqq y_i\leqq d$ to the determination of an invariant element of a functional transformation after the manner of Birkhoff-Kellogg-Schauder. Here D is the symmetric derivative defined by

$$Dz=\lim_{h\to 0}(z(x+h, y+h)-z(x-h, y+h)-z(x+h, y-h)+z(x-h, y-h))/(4h^2).$$

T. H. Hildebrandt (Ann Arbor, Mich.).

Calculus of Variations

***Bliss, Gilbert A. Lectures on the Calculus of Variations.** University of Chicago Press, Chicago, Ill., 1946. ix+296 pp. $5.00.

This is a sound, thorough and up-to-date text on the single integral problems of the calculus of variations, based on courses given by the author at the University of Chicago. The book is in two parts, part I containing six chapters on the simpler problems of the calculus of variations without side conditions, while part II consists of three chapters on the problem of Bolza. There is also an appendix on existence theorems for implicit functions and differential equations, a bibliography for the problem of Bolza, and an index.

The book starts with a typical simple problem, the non-parametric problem in 3-space with fixed end-points. Chapter I takes up necessary conditions, chapter II sufficient conditions for this case. Tables show various ways the necessary conditions of Euler, Weierstrass, Legendre and Jacobi may be strengthened to give sufficient conditions. In addition to the usual type of proof involving the Hilbert invariant integral and the construction of fields, sufficiency conditions without the use of fields are included.

Chapter III on fields and the Hamilton-Jacobi theory is useful for applications both to classical mechanics and quantum mechanics. It deals with canonical variables, the canonical equations, transversal surfaces and the Hamilton-Jacobi partial differential equations. There is also a discussion of Hamilton's principle and of Carathéodory's treatment of extremals as curves of quickest descent. Chapter IV gives the minor modifications required for problems in the plane and in n-space, rather than in 3-space. Chapter V deals with problems in parametric form. Chapter VI, the last chapter of part I, is concerned with problems with one or both end-points variable. Here transversality conditions must be satisfied, and the Jacobi condition is replaced by a more complicated fourth necessary condition, in deriving which here, as elsewhere, good use is made of the method of the accessory minimum problem of minimizing the second variation.

Part II presents the first comprehensive treatment of the problem of Bolza, a very general type of problem with side conditions and variable end-points. The form in which the results are presented here is that preferred by Bliss himself. The problem has now reached a state of completeness and simplicity which allows it to be presented in a manner like that used for the older problems of part I. Chapter VII on the multiplier rule introduces the problem. The theoretical equivalence of the problems of Lagrange, Mayer, and Bolza with side conditions and variable end-points is shown. The Bolza formulation has the advantage that the accessory minimum problem is of the same form as the original problem. The principal theorems of this chapter are the imbedding lemma, the multiplier rule, the Euler-Lagrange equations, the Weierstrass-Erdmann corner condition, and the Hilbert differentiability condition. The chapter contains also a clear discussion of the meaning of the important notions of normality and degree of abnormality.

Chapter VIII takes up further necessary conditions, the conditions of Weierstrass and Clebsch, the second variation, the fourth necessary condition and the accessory minimum problem. The last chapter is on sufficient conditions for the problem of Bolza. This contains some of the most difficult material of the book. The proofs are based on a method given originally by Hestenes. They require that the minimizing arc be normal, but not that it be normal on every subinterval. The method applies also to some abnormal cases. The proof is first made for the simpler case of problems with separated end conditions and satisfying the nontangency condition. In this chapter Bliss also refers to the work of other writers, especially on the question of the extent to which conditions of normality can be eliminated from proofs of the various necessary and sufficient conditions.

The theory here presented marks the culmination of the modern phase of development of the calculus of variations, begun by Weierstrass and continued by Hilbert, Bolza and Bliss. In this treatment the subject is studied rigorously, with no emphasis on the formal manipulation of symbols. On the other hand the subject is taken as an end in itself, and not as a mere adjunct of mechanics. Some of the other modern developments, such as the direct methods of Tonelli and the calculus of variations in the large of Morse, are not included in this book. Multiple integral problems, in which progress has been slower, are also not treated.

When used as a text, this book should be supplemented by illustrative examples, only a few of which are given. In fact, very few examples have ever been worked out to illustrate the more recent theory. Examples for the simpler problems may be found in older texts. *O. Frink.*

Zimmerberg, Hyman J. A self-adjoint differential system of even order. Duke Math. J. 13, 411–417 (1946).

The author considers the self-adjoint boundary value problem

$$\sum_{i=0}^{n}[f_i(x)u^{(i)}]^{(i)}=\lambda\sum_{j=0}^{n-1}[g_j(x)u^{(j)}]^{(j)}, \qquad a\leqq x\leqq b,$$

$$U_k(u,\lambda)=U_k^0(u)+\lambda U_k^1(u)=0, \quad k=1,\cdots,2n,$$

in which $U_k(u,\lambda)$ are independent linear forms in the values of $u, u', \cdots, u^{(2n-1)}$ at $x=a$ and $x=b$ for which $U_k^1(u)$ involves only the end values of $u, u', \cdots, u^{(n-1)}$. He proves that this system is equivalent to the Euler-Lagrange equations and transversality conditions associated with the problem of minimizing an expression of the form

$$J_2(\eta)=2Q[\eta(a),\eta(b)]+\int_a^b 2\omega(x,\eta,\eta')dx,$$

subject to a set of the usual linear differential equations of the first order and the usual linear homogeneous end conditions, and in addition to an isoperimetric condition requiring an expression of the form

$$K_2(\eta)=2G[\eta(a),\eta(b)]+\int_a^b \eta_i K_{ij}(x)\eta_j dx$$

to be constant. Here Q and G are quadratic forms in the variables $\eta_i(a)$, $\eta_i(b)$ and 2ω is a quadratic form in η_i, η_i', $i=1,\cdots,n$. *J. E. Wilkins, Jr.* (Buffalo, N. Y.).

Faedo, Sandro. Il calcolo delle variazioni per gli integrali su un intervallo infinito. I. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 4, 207–212 (1943).

The author states some existence theorems for integrals of the calculus of variations extended over infinite intervals. Proofs are to appear elsewhere. *L. M. Graves.*

Faedo, Sandro. Il calcolo delle variazioni per gli integrali su un intervallo infinito. II. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 4, 299–303 (1943).

[Cf. the preceding review.] The author discusses some necessary conditions for a minimum of an integral extended over an infinite interval. *L. M. Graves* (Chicago, Ill.).

Sólyi, A. Über das Haarsche Lemma in der Variationsrechnung und seine Anwendungen. Acta Univ. Szeged. Sect. Sci. Math. 11, 1–16 (1946).

This paper discusses Haar's lemma for multiple integral problems, its converse, and applications to a simple case of variable boundaries and to the second variation. [Proofs under weaker restrictions have been previously given by other authors. See Morrey, Univ. Calif. Publ. Math. (N.S.) 1, 1–130 (1943), in particular, pp. 66–74; Carson, Contributions to the Calculus of Variations, 1938–1941, University of Chicago Press, 1942, pp. 453–489; Reid, Duke Math. J. 5, 856–870 (1939); these Rev. 6, 180; 4, 48; 1, 78.]

L. M. Graves (Chicago, Ill.).

Theory of Probability

Cattaneo, Paolo. Tre problemi sulle concordanze. Atti Mem. Accad. Sci. Padova. Mem. Cl. Sci. Fis.-Mat. (N.S.) 57, 139–148 (1941).

In a game of patience called counting by m's, numbers 1 to m are counted in sequence as cards are dealt and the

count repeated until the deck is fully dealt. Hits occur when the card dealt and the number called agree. This paper gives the probabilities for each number of hits, counting by twos, when the deck dealt has cards of 4 colors each of n cards. Counting by threes, probabilities are given for (i) a deck of 4 colors each of 3 cards and (ii) a deck of 4 colors each of 10 cards. In the last, because of the extensive calculations needed for the direct methods used, probabilities for 0, 11, and 12 hits only are given.

J. Riordan (New York, N. Y.).

Gennaro, Antonino. Eventi subordinati linearmente dipendenti. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **13**, 947–962 (1942).

Consider random events $A_1, \cdots, A_n$. The author proposes to find conditions under which the probability of A_n can be uniquely determined in terms of the remaining probabilities whenever the latter are prescribed in a consistent, but otherwise arbitrary, way. The problem is generalized to the case where the A_k are intersections with other events whose probabilities are also indeterminate. The problem is shown to be reducible to a geometric problem and is solved for $n=2, 3, 4$. *W. Feller* (Ithaca, N. Y.).

Geppert, Maria-Pia. Das Bayessche Rückschlussproblem. Deutsche Math. **7**, 1–22 (1942).

Expository paper. *J. L. Doob* (New York, N. Y.).

Nicolini, Tito. Un tipo di curva a curvatura distribuita come la densità della probabilità nella legge normale. Rend. Accad. Sci. Fis. Mat. Napoli (4) **13**, 109–115 (1945).

The familiar equations of a curve (in terms of the arc length s) such that the curvature at s equals a prescribed function $f(s)$ are applied to the case where $f(s)$ is the normal density function. A few coordinates are computed and plotted. *W. Feller* (Ithaca, N. Y.).

Beretta, L. A proposito del lemma di Bienaymé. Giorn. Ist. Ital. Attuari **12**, 122–126 (1941).

It is shown how Selberg's inequalities [Skand. Aktuarietidskr. **23**, 121–125 (1940); these Rev. **1**, 245] can be deduced, in a direct way, from an inequality of Cantelli [Atti Accad. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (5) **11**, 330–349 (1916), in particular, p. 333]. *W. Feller* (Ithaca, N. Y.).

Jordan, Charles. Complément au théorème de Simmons sur les probabilités. Acta Univ. Szeged. Sect. Sci. Math. **11**, 19–27 (1946).

Let $x_1, \cdots, x_n$ be independent observations on the chance variable X, which is such that $P\{X=1\}=p<\frac{1}{2}$, $P\{X=0\}=q=1-p$, and write $Z=\sum_{i=1}^{n}x_i$. Let $\epsilon=np-[np]$, consider p as fixed, and let n (and hence ϵ) vary. The author proves that there exists a number ϵ_0, $\frac{1}{2}<\epsilon_0<q$, such that, when $\epsilon<\epsilon_0$, $P\{Z>np\}<P\{Z<np\}$ and vice versa. He has not succeeded in determining the exact value of ϵ_0, but finds it to be approximately $\frac{1}{2}(q+\frac{1}{2})$. *J. Wolfowitz.*

Loève, Michel. Fonctions aléatoires à décomposition orthogonale exponentielle. Revue Sci. **84**, 159–162 (1946).

Let $X(\alpha)$ be a real chance variable for each α in an interval and suppose that $E\{X(\alpha)\}=0$, $E\{X(\alpha)\overline{X(\beta)}\}=\psi(\alpha+\beta)$, where ψ is continuous. Then $X(\alpha)$ is called exponentially convex. In this case, ψ is the bilateral Laplace Stieltjes transform of a monotone function; $X(\alpha)$ is analytic and can be written in the form (*) $X(\alpha)=\int_{-\infty}^{\infty}e^{\alpha t}d\xi(t)$, where the t-increments of $\xi(t)$ in nonoverlapping intervals are mutually orthogonal. If, however, ψ is a function of $\alpha\beta$, (*) becomes $X(\alpha)=\int_{-\infty}^{\infty}\alpha^t d\xi(t)$. These results are extended to the case of chance vectors, to obtain the analogue of Cramér's conditions for the correlation and spectral functions of stationary vector processes [Ann. of Math. (2) **41**, 215–230 (1940); these Rev. **1**, 150]. *J. L. Doob* (Urbana, Ill.).

Adler, H. A., and Miller, K. W. A new approach to probability problems in electrical engineering. Trans. Amer. Inst. Elec. Engrs. **65**, 630–632 (1946).

The reviewer was unable to interpret the problems of the paper in a way which would lead to the results as presented by the authors without proofs. The main theorem is as follows. "Assume that each of n events has a rate of occurrence $p=t/T$ so that each event occurs once in a period of T units of time and lasts each time t consecutive units of time. Further, be it assumed that the starting points of the events can change only in steps of multiples of the unit of time selected. Then the average duration, or overlap, of r simultaneous events out of n, expressed in the selected units of time" is

$$t_r=\frac{t^r(T-t)^{n-r}}{t^r(T-t)^{n-r}-(t-1)^r(T-t-1)^{n-r}}.$$

W. Feller (Ithaca, N. Y.).

Feller, W. The law of the iterated logarithm for identically distributed random variables. Ann. of Math. (2) **47**, 631–638 (1946).

Soit une suite de variables aléatoires $\{X_n\}$ mutuellement indépendantes, ayant la même fonction de répartion $F(x)$ telle que

$$\int_{-\infty}^{\infty}xdF(x)=0, \quad \int_{-\infty}^{\infty}x^2dF(x)=1.$$

Posons $S_n=\sum_{i=1}^{n}X_i$; une suite de nombres positifs monotone $\{\varphi_n\}$ est dite de la classe supérieure s'il est presque-certain que l'inégalité (1) $S_n>\varphi_n n^{\frac{1}{2}}$ n'est réalisée que pour un nombre fini de valeurs de n, et de la classe inférieure s'il est presque-certain que (1) est réalisée pour une infinité de valeurs de n. L'auteur démontre que si, lorsque $x\to\infty$, on a

$$(2) \qquad \int_{|t|>x}t^2dF(t)=O\{(\log\log x)^{-1}\},$$

une condition nécessaire et suffisante pour que $\{\varphi_n\}$ soit de la classe supérieure est que $\sum_n n^{-1}\varphi_n e^{-\frac{1}{2}\varphi_n^2}<\infty$. Ce résultat n'est plus valable si la condition (2) n'est pas réalisée.

R. Fortet (Caen).

Cheng, Tseng-Tung. On the combination of statistical elements. Coll. Papers Sci. Engin. Nat. Univ. Amoy **1**, 73–82 (1943).

To justify a vague statement in a textbook the author proves a special case of the central limit theorem. He evidently had no access to the literature. *W. Feller.*

von Schelling, Hermann. Das Alles- oder- Nichts-Gesetz, gedeutet als Endergebnis einer Auslösungsfolge. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1944**, no. 6, 25 pp. (1944).

The mathematical part of the paper deals with chain reactions of the following type. An initial element (person, gene, corpuscle) has probability p_k of creating k similar elements of the first generation. The process continues so that each element of the nth generation has the same proba-

bility p_k of producing k elements of the $(n+1)$th generation, and all elements of the same generation are statistically independent. According to Lotka [Théorie Analytique des Associations Biologiques, II, Actual. Sci. Indus. no. 780, Hermann, Paris, 1939, pp. 123 ff.], this process occupied H. W. Watson and F. Galton before 1889 (in connection with the survival of family names). Since then it has been treated by many authors; the best known treatment is that by R. A. Fisher [The Genetical Theory of Natural Selection, Oxford University Press, 1929, pp. 273 ff.]. The special case $p_N=\zeta$, $p_0=1-\zeta$ has recently been discussed by Schrödinger [Proc. Roy. Irish Acad. Sect. A. 51, 1–8 (1945); these Rev. 7, 457]. It turns out in general that either the progeny increases indefinitely or else the process stops. This well-known result is verified by the author for the particular case $p_k=e^{-a}a^k/k!$. It is said to be novel and is regarded as a statistical proof of the "everything-or-nothing law," which is not formulated since it is regarded as an obvious and well-known fact. The author tries to force the problem by computations. Accordingly, he struggles unsuccessfully with the case of a Pólya distribution and claims that the verification of the result in more general cases cannot be expected. This claim is partially contradicted by the computations of the last section, which was clearly added in proofs. Applicability of the theory to empirical phenomena is claimed, in particular, to epidemics. How the fact that the entire population is not infected follows from the "everything-or-nothing law" is not explained.

W. Feller (Ithaca, N. Y.).

Bochner, S. Finitely additive set functions and stochastic processes. Proc. Nat. Acad. Sci. U. S. A. 32, 259–261 (1946).

The paper is a summary of results to be published in full elsewhere. A function $f(E, \lambda)$ defined for λ in a measure space $\{\lambda\}$ of total measure 1 and a complemented lattice B of subsets E of a set S is called a stochastic phenomenon if it is finitely additive for disjoint sets $E_1, \cdots, E_n$. A partition δ is a finite system of disjoint $E_1, \cdots, E_n$, and $\varphi_\delta(d_1, \cdots, d_n)$ is the characteristic function of the n distribution functions of the variables $f(E_k, \lambda)$. The relation between $\phi_\delta(\alpha)$ and $\phi_{\delta'}(\alpha')$ when δ' is a sub-partition of δ can be expressed by consistency conditions $\alpha_k' = \sum_n c_{kn}\alpha_n$ and, conversely, if there exists a consistent system of characteristic functions for a directed set of partitions $\{\delta\}$, a compatible stochastic phenomenon can be constructed. If a measure $\mu(E)$ exists on B, it is possible to define integrals $\zeta(\lambda) = \int_S h(x)df(E, \lambda)$ and to consider discrete expansions in orthogonal functions $h_n(x)$ of the form

$$\frac{df(x, \lambda)}{dx} \sim \sum f_n(\lambda)h_n(x), \qquad f_n(\lambda) = \int h_n df.$$

As a particular case of the continuous analogue of this expansion, it is stated that Wiener differential process defined by $\phi_\delta(\alpha) = \prod \exp(-\mu(E_\nu)\Psi(\alpha_\nu))$, $\Psi(\alpha) = \alpha^2$, is its own sine or cosine transformation in $(0, \infty)$. *H. R. Pitt.*

Blackwell, D., and Girshick, M. A. On functions of sequences of independent chance vectors with applications to the problem of the "random walk" in k dimensions. Ann. Math. Statistics 17, 310–317 (1946).

Soit $X_1, \cdots, X_n, \cdots$ une suite indéfinie de vecteurs aléatoires indépendants dans un espace à k dimensions; soit S_n $(n=1, 2, \cdots)$ un événement aléatoire fonction de $X_1, \cdots, X_n$, de telle sorte que les S_n soient mutuellement incompatibles et que la somme de leurs probabilités est égale à 1; soit φ_n $(n=1, 2, \cdots)$ une variable aléatoire réelle ou complexe, fonction de $X_1, \cdots, X_n$, telle que $E(\varphi_n)=0$ et que la moyenne conditionnelle $E(\varphi_n/X_1, \cdots, X_m)$ soit égale à φ_m pour tout $n \geqq m$. Soit N la variable aléatoire qui prend la valeur n si S_n a lieu et φ la variable aléatoire φ_N; en supposant que $E(N)$ est finie, les auteurs démontrent que, s'il existe une fonction (numérique) $f(X)$ du vecteur X, non négative, telle que $E[f(X_n)] < \infty$ et que $|\varphi_n| \leqq \sum_{i=1}^{n} f(X_i)$ lorsque $N \geqq n$, $E(\varphi)$ existe et est égale à 0. Ce théorème résoud une question posée par la théorie des tests en chaîne (sequential tests); les auteurs en déduisent des généralisations de résultats déjà obtenus par Wald et par eux-mêmes, et une application au problème de la "promenade au hasard" dans un espace à k dimensions. *R. Fortet* (Caen).

Schulz, Günther. Über die Häufigkeit der Iterationen in einer Beobachtungsfolge. Deutsche Math. 7, 22–38 (1942).

A sequence of chance variables $x_0, \cdots, x_n$ which form a Markov chain is defined as follows. Every x can take only the integral values $1, \cdots, k$. Let $P\{x_j=i\} = q_i^{(j)}$. The sequence $q_1^{(0)}, \cdots, q_k^{(0)}$ is given. Also given is the matrix $V=\{v_{it}\}$ of transition probabilities, where $q_i^{(j)} = \sum_{t=1}^{k} q_t^{(j-1)} v_{it}$. It is assumed that the matrix V is regular in the sense of Hadamard and Fréchet (i.e., each of the $q_i^{(j)}$ approaches a limit as $j \to \infty$). Let y be the number of runs (iterations) of the same elements in the sequence $x_1, \cdots, x_n$. It is proved that, with the usual normalization, y is asymptotically (with n) normally distributed, with given mean and variance. Application is made to the sex distribution of births, where the lack of independence is considered to be due to the fact that twins are more often than not of the same sex. Also let z_{st} be the number of observations t immediately preceded by s $(s, t=1, \cdots, k)$ in the sequence $x_1, \cdots, x_n$. It is shown that z_{st} is asymptotically normal, with the usual normalization. The mean and variance are given and various special cases discussed. The paper closes with remarks about, and a list of errata for, an earlier paper of the author [Deutsche Math. 1, 665–699 (1936)]. *J. Wolfowitz.*

Rutman, M. A. Concerning a paper by T. A. Sarymsakov. C. R. (Doklady) Acad. Sci. URSS (N.S.). 52, 567–568 (1946).

Sarymsakov's paper referred to in the title appeared in the same C. R. 49, 85–88 (1945) [these Rev. 7, 459]. The author shows by means of a simple counterexample that the condition $K^{(m)}(x, x) > 0$ of the main theorem (quoted in the review) is not necessary. It is shown that the sufficiency of the condition is a simple consequence of a theorem of the author [same C. R. (N.S.) 18, 625–627 (1938); Rec. Math. [Mat. Sbornik] N.S. 8(50), 77–96 (1940); these Rev. 2, 104]. *W. Feller* (Ithaca, N. Y.).

Cantelli, F. P. I fondamenti matematici della tecnica delle assicurazioni. Giorn. Ist. Ital. Attuari 13, 1–27 (1942).

A lecture on various applications of the theory of summation of random variables to problems in risk theory. The author is led, in particular, to the compound Poisson process [cf., e.g., A. Khintchine, Asymptotische Gesetze der Wahrscheinlichkeitsrechnung, Ergebnisse der Math., v. 2, no. 4, Springer, Berlin, 1933, p. 24]. *W. Feller* (Ithaca, N. Y.).

Segerdahl, C.-O. Über einige risikotheoretische Fragestellungen. Skand. Aktuarietidskr. 25, 43–83 (1942). [MF 14136]

The collective risk-theory of F. Lundberg has been criticized by G. Ottaviani [Giorn. Ist. Ital. Attuari 11, 163–189

(1940)] who, among other objections, stated that this theory disregards the effect of the interest rate on the risk-reserve. The present paper contains a brief recapitulation of the collective risk-theory, with emphasis on the refutation of Ottaviani's criticisms. In the concluding chapter an extension of the collective risk-theory is indicated which takes into account the effect of interest.

Z. W. Birnbaum (Seattle, Wash.).

Blanc-Lapierre, André. Étude des fluctuations produites par l'effet de grenaille dans les amplificateurs. Revue Sci. 84, 75–94 (1946).

Expository article.

Blanc-Lapierre, A., et Lapostolle, P. Fluctuations dans les grandeurs physiques quasi sinusoïdales. J. Phys. Radium (8) 7, 153–164 (1946).

Extension of remarks in C. R. Acad. Sci. Paris 222, 1324–1325 (1946); these Rev. 7, 460. *J. L. Doob.*

Frenkiel, François-N. Étude statistique de la turbulence: théorie de la mesure de la corrélation avec deux fils chauds non compensés. C. R. Acad. Sci. Paris 222, 1377–1378 (1946).

As in a previous paper [same C. R. 222, 585–587 (1946); these Rev. 7, 461], the author studies the statistical properties of turbulent velocities if one assumes that the measurements are done by a hot wire anemometer, without an electrical network to compensate for the lag of the wire. He gives the relation between the correlation function computed with that assumption and its true value.

J. Kampé de Fériet (Lille).

Frenkiel, François-N. Étude statistique de la turbulence: théorie de la mesure de l'intensité de la turbulence avec un fil chaud de longueur non négligeable. C. R. Acad. Sci. Paris 222, 1474–1476 (1946).

The effect of the length l of the wire on the measurements of turbulent velocities by a hot wire anemometer has been studied by H. L. Dryden, G. B. Schubauer, W. C. Mock and H. K. Skramstad [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 581 (1937)]; the author gives a simpler way of getting some of their results and some new relations; one of these is

$$E/Eb = l \Big/ \int_0^l R_x(s)ds,$$

E and Eb being, respectively, the true and the measured values of the kinetic energy of the turbulent longitudinal velocities and R_x the correlation function of G. I. Taylor.

J. Kampé de Fériet (Lille).

Mathematical Economics

Bentzel, R., and Wold, H. On statistical demand analysis from the viewpoint of simultaneous equations. Skand. Aktuarietidskr. 29, 95–114 (1946).

Consider a system of linear stochastic difference equations

$$(1) \qquad \alpha_g + \sum_{h=1}^{H} \sum_{\tau=0}^{\tau_{gh}} \alpha_{ghr} x_h(t-\tau) = u_g(t), \qquad g = 1, \cdots, G \leqq H;\ t = 1, \cdots, T,$$

where t is the time of observations on the x's and the u's are nonobservable jointly normally distributed "disturbances." The τ's are time-lags. H. B. Mann and A. Wald [Econometrica 11, 173–220 (1943), in particular, pp. 192 ff.; these Rev. 5, 129] have proved that the maximum likelihood estimates of the α's and σ's are asymptotically unbiased, in the case $G=H$, $\mathcal{E}u_g(t)=0$, $\mathcal{E}u_g(t)u_h(t')=\sigma_{gh}\delta_{tt'}$. Suppose in addition that the subscripts in (1) can be arranged so as to make (2) $\alpha_{ghr}=0$ for $g<h$ and suppose that (3) $\mathcal{E}u_g(t)u_h(t')=\sigma_g{}^2\delta_{gh}\delta_{tt'}$. We can always make $\alpha_{gg0}=1$ for all values of g. Then, owing to (2), the Jacobian of the transformation carrying the probability density function from the u-space into the x-space will equal 1. Maximizing the resulting likelihood function jointly with respect to $\sigma_g{}^2$ and to the unknown α's entering the gth equation in (1) we obtain estimates which, because of (3), are identical with the least-squares estimates of the regression coefficients of $-x_g(t)$ on $x_{g-1}(t), x_{g-2}(t), \cdots, x_g(t-1), x_{g-1}(t-1), x_{g-2}(t-1), \cdots$. The proof was outlined by T. Haavelmo [Econometrica 11, 1–12 (1943), in particular, § 3, last paragraph; these Rev. 4, 220]; and, with reference to a simple economic example ($G=2$), by T. Koopmans [J. Amer. Statist. Assoc. 40, 448–466 (1945), in particular, pp. 459–461; these Rev. 7, 215]. Bentzel and Wold think that (2), (3) are often satisfied by market models (1) which describe the behavior of buyers and sellers; they give an example.

They also study the case when some of the lags τ are shorter than the distance between successive observations. Let some of the $x(t-\tau)$ be approximated by interpolation; the letters x, u, α in (1) must then be replaced by x^*, u^*, α^*, say; the condition (2), if satisfied by α, will, in general, not be satisfied by the approximate parameters α^*. Therefore the least-squares method applied to the time series x^* will not give consistent estimates of either α or α^*.

Finally, the authors point out a statement by S. Malmquist [unpublished] that, if the variables are connected by nonlinear relations, the replacing of a sequence of values of a variable by its integral over time leads to an error in the estimates of the parameters of these relations.

J. Marschak (Chicago, Ill.).

Amoroso, L. La trasformazione di valore nel processo produttivo. Giorn. Ist. Ital. Attuari 11, 89–100 (1940). [MF 16624]

Let vectors q and p denote the prices and x and z the quantities of, respectively, a firm's inputs and outputs (of various kinds). Then $z=z(x, \dot{x})$ is a functional of $x(t)$, where t denotes time and $\dot{x} \equiv dx/dt$. Assume that (I) the firm maximizes $\int_{t_0}^{t_1}(T-qx)dt$, where $T=pz$, (II) boundary conditions are given either by fixing $x(t_0)$ and $\dot{x}(t_0)$ ("evolution") or by fixing $x(t_0)$ and $x(t_1)$ ("revolution"), (III) p, q are constant and the interest rate $i=0$. Euler's condition yields

$$(A) \qquad \frac{\partial T}{\partial x} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - q = 0.$$

Traditional (static) economics ignored the middle term which the author likens to inertia in mechanics. He makes further analogies. Let the firm's unused capacity be $V \equiv q \cdot (x_{\max} - x)$; it is likened to potential energy. The analogue U of kinetic energy is defined by the energy conservation principle, $\dot{U}+\dot{V}=0$. Hence $\dot{U}=q\dot{x}$. Using (A), U can be expressed in terms of T, $\dot{x}$, $\partial T/\partial \dot{x}$ and $\ddot{x}$ [the term involving $\ddot{x}$ is omitted from the author's formula (16) for U by an oversight in deriving (13) from (5)]. When $\dot{x}$ (and $\ddot{x}$) vanish, $T=U$; therefore U is also called "latent production." When condition (III) is relaxed by making p and q functions of t and by allowing $i>0$, the energy conservation principle is preserved by introducing a correction term which the author

calls "monetary energy" and in which he sees some analogies with heat. He points out that in economics the principle of least action is a fundamental postulate, not, as in physics, an implication derived from other principles. [There is an English version of the article [Econometrica 8, 1–11 (1940); these Rev. 1, 155]. Earlier applications of the calculus of variations to the dynamics of a firm are due to Evans [Amer. Math. Monthly 31, 77–83 (1925)] and Roos [Amer. J. Math. 47, 163–175 (1925)]; for further development and bibliography see G. C. Evans, Mathematical Introduction to Economics, McGraw-Hill, New York, 1930, chaps. XIV–XV and appendix II.] *J. Marschak* (Chicago, Ill.).

***Roy, René. De l'utilité. Contribution à la théorie des choix.** Actualités Sci. Ind., no. 930. Hermann et Cie., Paris, 1942. 47 pp.

This article is concerned with certain transformations on the traditional utility functions. It is claimed that the results are more convenient to statistical observation. The usefulness of the results hinges on the possibility of an experimental determination of the indifference surfaces $S:U(\bar{Q})=C$, where $\bar{Q}$ is the quantity vector whose components are the individual quantities of goods constituting an equilibrium complex. Given a system of prices defined by the vector $\bar{P}$ and the money income of an individual r, the consumer will distribute his purchases to maximize satisfactions. The vector $\bar{Q}$ adapted to $\bar{P}$ and r is defined by the tangency of the budget plane $r=\sum pq$ and an indifference surface. Subject to the condition that Q remains an equilibrium complex, total utility can be written as $U=\phi(\bar{P},r)=C$. This is the tangential equation of S. From this it is easy to derive the equilibrium equations in tangential coordinates, $\phi_1/q_1=\cdots=\phi_n/q_n=-\phi_r$, where the ϕ_i are the partial derivatives of $\phi(\bar{P},r)$ with respect to the n individual prices and the money income r.

Let $\bar{P}_0$ and $\bar{P}$ represent the base-reference and current period price vectors. A price index z, based on the notion of equivalent Q, is defined by $z=\bar{Z}/\bar{P}_0$, where $U(\bar{Z},r)=U(\bar{P},r)$. The total utility, under these restrictions, can be represented as a function of the real income $\rho=r/z$. For the general case the differential forms of the indexes of price and real income are $dz/z=\sum qdp/\sum pq$ and $d\rho/\rho=\sum pdq/\sum pq$. The indexes are applicable only to the individual consumer or homogeneous social groups. *M. P. Stoltz.*

GEOMETRY

Bachmann, Friedrich. Ein lineares Vollständigkeitsaxiom. Jber. Deutsch. Math. Verein. 53, 49–56 (italic) (1943).

As the capstone of his system of axioms for Euclidean geometry, Hilbert [Grundlangen der Geometrie, 7th ed., Teubner, Leipzig-Berlin, 1930] formulated a "completeness axiom" for the purpose of insuring that the only geometries isomorphic with ordinary Cartesian geometry are those which satisfy the entire set of axioms [I (1–8) Connection, II (1–4) Order, III (1–5) Congruence, IV Parallel, V (1–2) Continuity (V1, Archimedean; V2, Completeness)]. In earlier editions the completeness axiom took the form of precluding the extension of the system of points, lines and planes so that all the previous axioms would be valid in the extended system. In the 7th edition Hilbert showed that Cartesian geometry may be characterized by axioms I–IV, V1, together with an axiom of linear completeness which postulates that no extension is possible by which at least one point is added to the points of a line, with axioms I (1–2), II, III, V1 valid for the extended line. Since the axiom groups II, III contain axioms referring to the plane, the question arose whether the desired characterization might be obtained by adjoining to axioms I–IV, V1 an axiom V2 of linear completeness in the strong sense which, like the above, refers only to extension of the points of a line, but (unlike the above) demands the maintenance of linear properties only. The principal result of the paper consists in proving that such a linear completeness axiom in the strong sense is the following. It is not possible to extend the system of points on a line so that in the extension linear order, linear congruence axioms (including the uniqueness of segment transportation) and the Archimedean axiom are all valid. In the Hilbert formulation, this demands the continued validity of axioms II (1, 3), theorem 5, axiom III (1–3) (it is seen that III1 may be suppressed) and V1.

L. M. Blumenthal (Columbia, Mo.).

Kabaker, Nathan. Note on the importance of the axiom of Pasch in Euclidean geometry. Riveon Lematematika 1, 28 (1946). (Hebrew)

Wunderlich, Walter. Darbouxsche Verwandtschaft und Spiegelung an Flächen 2. Grades. Deutsche Math. 7, 417–432 (1944).

The two most important Euclidean models for the non-Euclidean plane are the projective model of Beltrami and Cayley, where lines remain straight, and the conformal model of Klein and Poincaré, where lines become circles. The connection between them is described as follows. Given a sphere σ and the plane π of a small circle, let AA' be the diameter of σ perpendicular to π, and let P be any point on π, within σ. The line through P parallel to AA' meets σ in two points Q which are projected from A into two points P^* on π. Then P and P^* (the orthogonal and stereographic projections of Q) are the projective and conformal representatives of one point of the hyperbolic plane. For the analogous treatment of the elliptic plane, let P be any point on π (possibly outside σ). The two ends Q of the diameter through P are projected from A into two points P^* on π. Then P and P^* (the gnomonic and stereographic projections of Q) are the projective and conformal representatives of one point of the elliptic plane. The author investigates the "Darboux transformation" from the one point P to the two points P^*, showing how it may be obtained by reflection in a quadric of revolution. *H. S. M. Coxeter.*

v. Szász, Paul. Absonderung der elliptischen, Euklidischen und hyperbolischen Geometrie. Mat. Fiz. Lapok 48, 243–271 (1941). (Hungarian. German summary)

In der vorliegenden Arbeit wird von neuem ein Axiomensystem der Raumgeometrie aufgestellt, aus welchem der Reihe nach die elliptische, die Euklidische, die hyperbolische (Bolyai-Lobatschefskijsche) Geometrie folgt, wenn man nur noch das entsprechende Parallelenaxiom für eine einzige Gerade und für einen einzigen ausser ihr gelegenen Punkt voraussetzt. Die gewählten Axiome beziehen sich unmittelbar auf den gesamten Raum, und die Absonderung der drei Geometrien (der Beweis des unten stehenden Satzes) geschieht ohne Benützung der Stetigkeit.

Zunächst wird auf Grund der gemeinsamen Verknüpfungs-, Anordnungs- und Kongruenzaxiome (die naturgemäss

von der Hypothese der offenen oder geschlossenen Gerade unabhängig sind) der bekannte allgemeine Satz bewiesen: Unter den Geraden, die in einer Ebene (g, P) durch den Punkt P gezogen werden, gibt es entweder keine, oder eine und nur eine, oder aber mehrere, die die Gerade g nicht schneiden, je nachdem der erste, zweite oder dritte Fall für eine einzige Gerade g_0 und für einen einzigen ausser ihr gelegenen Punkt P_0 vorliegt.

Weiter wird gezeigt, dass die drei verschiedenen Parallelenaxiome, die das Eintreten des ersten, zweiten, bzw. dritten Falles bestimmen, zusammen mit den obigen und mit dem Dedekindschen Stetigkeitsaxiom, je ein vollständiges Axiomensystem der elliptischen, Euklidischen, bzw. hyperbolischen Geometrie bilden. *Author's summary.*

v. Szász, Paul. Über die hyperbolische Trigonometrie. Mat. Fiz. Lapok **48**, 401–409 (1941). (Hungarian. German summary)

This is a new derivation of the classical formulas connecting the sides and angles of the general hyperbolic triangle. It shows no marked advantage over the familiar treatments. *H. S. M. Coxeter* (Notre Dame, Ind.).

Rauter, Herbert. Die Extremwertaufgabe von Regiomontan in der hyperbolischen Geometrie. Deutsche Math. **7**, 378–382 (1944).

Let O, A, B be three fixed collinear points, and M a variable point on the perpendicular line OX. Regiomontanus (1436–1476) remarked that the maximum value of the angle AMB occurs when M is the point of contact of a circle through A and B that touches OX. The geometry was then, of course, Euclidean. The author seeks the point M from which the segment AB subtends the maximum angle in hyperbolic geometry. Since the locus of a point from which AB subtends a constant angle is no longer a circle, the desired point will not be given by the same construction as in the Euclidean case; so the author's numerical verification of the failure seems superfluous.

Let A, B, M be distant a, b, x from O. The angles $\alpha = OMA$ and $\beta = OMB$ are given by $\tan\alpha = \tanh a/\sinh x$, $\tan\beta = \tanh b/\sinh x$. The difference $\beta - \alpha$ will be maximum when $\sinh x + \tanh a \tanh b/\sinh x$ is minimum, i.e., when $\sinh x = (\tanh a \tanh b)^{\frac{1}{2}}$. The author missed this simple solution by employing a Euclidean model instead of working entirely in hyperbolic geometry. *H. S. M. Coxeter.*

Rauter, Herbert. Eine räumliche Weiterführung der Extremwertaufgabe von Regiomontan. Deutsche Math. **7**, 373–377 (1944).

Let ABC be a plane triangle, M the centre of a unit sphere, and $A'B'C'$ the spherical triangle formed by the intersections of the planes MAB, MBC, MCA with the surface of the sphere. The sum of the lengths of the sides of $A'B'C'$ is called the optical perimeter of ABC. The author considers maximum and minimum values of this perimeter. If, for example, M has the coordinates $(x, 0, 0)$, and the plane triangle is right-angled isosceles with vertices $A(0, -1, h)$, $B(0, 1, h)$, $C(0, 0, h+1)$, he proves that in the intervals $h \geqq -1$ and $h \leqq -2-3^{\frac{1}{2}}$ the optical perimeter varies continuously from an absolute minimum for $x = \infty$ to an absolute maximum at $x = 0$. In the interval $-2-3^{\frac{1}{2}} < h < -1$, however, the optical perimeter attains the maximum before $x = 0$ and tends to a relative minimum at $x = 0$. Results are also derived for an equilateral triangle with vertices $A(0, -1, h)$, $B(0, 1, h)$, $C(0, 0, h+3^{\frac{1}{2}})$. *W. H. Gage.*

Rademacher, Hans, and Toeplitz, Otto. The necessity of a compass in elementary geometrical constructions. Riveon Lematematika **1**, 14–19 (1946). (Hebrew)

Translation of § 21 [pp. 141–151] of the authors' "Von Zahlen und Figuren" [Springer, Berlin, 1930].

Wiedemann, B. Algebraisch-geometrische Untersuchungen über Konstruktionsmöglichkeiten auf der Kugel. Deutsche Math. **7**, 178–184 (1943).

The author adds some further constructions to his earlier work [Deutsche Math. **2**, 520–544 (1937)] on the extension of the Mohr-Mascheroni "geometry of compasses" from the plane to the sphere. *H. S. M. Coxeter.*

Lebesgue, Henri. Sur l'équivalence des polyèdres. Ann. Soc. Polon. Math. **18**, 1–3 (1945).

The note is a complement to a previous paper by the author [same Ann. **17**, 193–226 (1938)]. After a modification of the definition of the s_n and σ_n formula (3) on page 218 is proved in detail. *B. Jessen* (Copenhagen).

Hajós, G. Einfache Bedeckung mehrdimensionaler Räume mit Würfelgitter. Mat. Fiz. Lapok **48**, 37–64 (1941). (Hungarian. German summary)

A German version of this paper has already been reviewed [Math. Z. **47**, 427–467 (1941); these Rev. **3**, 302]. *P. Erdös* (Syracuse, N. Y.).

Popoviciu, Tiberiu. On regular polygons. Pozitiva **2**, 92–97 (1941). (Romanian)

Let $A_0, \cdots, A_{n-1}$ be the vertices of a regular $n$gon, P any point of its plane. Let $M_r(P)$ be the rth root of the arithmetic mean of the rth powers of the distances from P to $A_0, \cdots, A_{n-1}$. The author determines bounds for for $M_r(P)/M_s(P)$ for various values of r and s. In particular, $M_1(P)/M_2(P) \geqq 2^{\frac{1}{2}} n^{-1} \cot(\frac{1}{2}\pi/n)$. *R. P. Boas, Jr.*

Sas, E. On a certain extremum-property of the ellipse. Mat. Fiz. Lapok **48**, 533–542 (1941). (Hungarian. English summary)

The author proves that in any convex curve of area 1 one can inscribe an $n$gon of area at least $(\frac{1}{2}n/\pi)\sin(2\pi/n)$, with equality only for the ellipse. Also in any convex curve of area 1 one can inscribe an $n$gon of circumference at least $(n/\pi)\sin(2\pi/n)$, with equality only for the circle. *P. Erdös* (Syracuse, N. Y.).

Molnár, Josef. Über eine elementargeometrische Extremalaufgabe. Mat. Fiz. Lapok **49**, 249–253 (1942).

The author proves the following results. Denote by d the circle of greatest radius which can be covered by 3 circles of radii d_1, d_2, d_3. If the triangle determined by d_1, d_2 and d_3 is acute angled, d equals the radius of the circumscribed circle. If it is not acute angled $d = \max(d_1, d_2, d_3)$.

In the case of 4 circles we construct a quadrilateral inscribed in a circle from d_1, d_2, d_3, d_4. If the center is in the interior of the polygon, d equals the radius of the circle; if not, $d = \max(d_1, d_2, d_3, d_4)$. For 5 or more circles the problem is unsolved. Covering of spheres is also investigated. *P. Erdös* (Syracuse, N. Y.).

Fejes, Ladislaus. Das gleichseitige Dreiecksgitter als Lösung von Extremalaufgaben. Mat. Fiz. Lapok **49**, 238–248 (1942). (Hungarian. German summary)

The author investigates the following problem. Let there be given n circles of unit radius. What is the radius of the greatest circle which can be covered by these n circles? The author assumes that no three circles have an interior point

in common and determines the asymptotic value of the radius for large n; this is obtained if the n points form a triangular lattice. The assumption that no three circles have a common interior point is probably superfluous.

P. Erdös (Syracuse, N. Y.).

Fejes, László. Über die Bedeckung einer Kugelfläche durch kongruente Kugelkalotten. Mat. Fiz. Lapok **50**, 40–46 (1943). (Hungarian. German summary)

The author extends the result of the paper reviewed above to the general case. He points out that Kershner previously proved this by different methods [Amer. J. Math. **61**, 665–671 (1939); these Rev. **1**, 8]. He then proves some related more general theorems. *P. Erdös* (Syracuse, N. Y.).

Kilczer, Julius. Die scheinbare Bewegung. Mat. Fiz. Lapok **49**, 170–186 (1943). (Hungarian. German summary)

Es wird der Unterschied und Zusammenhang der kinematisch beschriebenen und der "scheinbaren" Bewegung am Beispiel der während des Reisens beobachtbaren scheinbaren Bewegung der Landschaft mit Rücksicht auf die Perspektive und auf die Grenze der Tiefenwahrnehmung erörtert. Es werden mit Hilfe der Ruletten zweier ausgewählter Punkte Basis- und rollende-Kurve der die relativen Bahnen der Punkte ergebenden ebenen Bewegung bestimmt.

Author's summary.

Lage Sundet, Knut. On Simson's theorem. Norsk Mat. Tidsskr. **27**, 112–114 (1945). (Norwegian)

Klug, Leopold. Konjugierte Kegelschnitt-Tripel und ihre speziellen Fälle. Mat. Fiz. Lapok **48**, 144–161 (1941). (Hungarian. German summary)

Kárteszi, F. Über das System der gleichseitigen Hyperbeln, die eine Parabel hyperoskulieren. Mat. Fiz. Lapok **48**, 193–202 (1941). (Hungarian. German summary)

[Cf. Boll. Un. Mat. Ital. (2) **2**, 314–320 (1940); these Rev. **2**, 78.] In einer früheren Arbeit habe ich das System jener Parabeln untersucht, die eine gleichseitige Hyperbel hyperoskulieren. In vorliegender Arbeit stelle ich die Frage umgekehrt: es sei das im Titel gekennzeichnete Hyperbelsystem zu untersuchen. *From the author's summary.*

Zacharias, Max. Untersuchungen über ebene Konfigurationen $(12_4, 16_3)$. Deutsche Math. **6**, 147–170 (1941).

The author showed in an earlier paper [Monatsh. Math. Phys. **44**, 153–158 (1936)] that Morley's two equilateral triangles (formed by the intersections of the trisectors of the angles of a given triangle) are in perspective with the given triangle and with each other, the three centers of perspective being collinear, so that the nine vertices of the three triangles and the three centers of perspective form a $(12_4, 16_3)$ configuration: 12 points lying by 3's on 16 lines, 4 through each point. At about the same time, J. M. Feld [Amer. Math. Monthly **43**, 549–555 (1936)] obtained another such configuration by taking the three real inflections of a plane cubic curve along with the nine real sextactic points. The author exhibits both of these as special cases of Hesse's $(12_4, 16_3)$, formed by the points of contact of the twelve tangents to a cubic that pass by fours through any three collinear points of the curve. He also exhibits Hesse's configuration as a projection of Stephanos' configuration of three desmic tetrahedra [Reye, Acta Math. **1**, 97–108 (1882)]. He then describes an entirely different $(12_4, 16_3)$ configuration, whose sixteen lines form four complete quadrilaterals, their vertices coinciding in pairs at the twelve points of the configuration. He shows that fifteen such configurations occur in the Pascal-Steiner hexagramma mysticum. *H. S. M. Coxeter* (Notre Dame, Ind.).

Villa, Mario. Trasformazioni quadratiche osculatrici ad una corrispondenza puntuale fra piani proiettivi. I. Le proiettività caratteristiche. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 718–724 (1942).

Villa, Mario. Trasformazioni quadratiche osculatrici ad una corrispondenza puntuale fra piani proiettivi. II. Loro costruzione. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 1–7 (1943).

L'emploi systématique de la variété de Segre V_4^6 de l'espace S_8, qui représente les couples de points de deux plans, permet d'étudier l'empreinte d'une transformation ponctuelle T sur le voisinage du second ordre d'un couple de points correspondants O, O'. Sur chacune des trois directions caractéristiques (inflexionnelles), T empreint, pour le voisinage du second ordre, une projectivité caractéristique, qui coïncide avec l'empreinte totale, sur cette droite, de l'une quelconque des transformations quadratiques qui osculent T.

Les 3 couples de directions inflexionnelles et les 3 projectivités caractéristiques déterminent complètement l'empreinte de T sur le voisinage du 2e ordre. Les points singuliers des transformations quadratiques osculatrices sont sur les droites inflexionnelles; le choix arbitraire de deux d'entre eux détermine univoquement le troisième, et le triangle singulier qui leur correspond dans l'autre plan. Il n'y a pas d'invariant projectif pour l'empreinte du second ordre de T. Parmi les ∞^2 projectivités tangentes à T (même empreinte sur le voisinage du 1er ordre), il existe 3 projectivités tangentes caractéristiques, dont chacune empreint sur 2 couples de droites inflexionnelles les projectivités caractéristiques. Il n'existe pas de transformation de De Jonquières (ni, en particulier, de transformation cubique) qui puisse approximer T jusqu'au voisinage du 3e ordre du couple O, O'.

P. Belgodère (Paris).

Bompiani, Enrico. Sulle trasformazioni puntuali fra piani proiettivi. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **13**, 837–848 (1942).

La considération des points situés sur les droites inflexionnelles établit, directement, des théorèmes découverts par M. Viala pour le voisinage du second ordre d'un couple de points correspondants. Chacune des ∞^2 transformations quadratiques osculatrices possède 4 directions d'hyperosculation (voisinage du 3e ordre), engendrant un système linéaire. Par rapport à chacun des 3 groupes de 2 directions inflexionnelles d'un même plan, il existe une et une seule transformation quadratique osculatrice T, canonique, pour laquelle les 4 directions d'hyperosculation se partagent en 2 groupes séparés harmoniquement par ce couple. Cela fixe un repère projectif privilégié, et 6 invariants projectifs, caractérisant le voisinage du 3e ordre de T, et pouvant s'interpréter par 6 correspondances (1, 2) sur les trois couples de droites inflexionnelles. *P. Belgodère* (Paris).

Bompiani, Enrico. Corrispondenza puntuale fra piani proiettivi: esame delle jacobiane. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **14**, 11–21 (1943).

On peut approximer une correspondance ponctuelle T entre deux plans projectifs, jusqu'au voisinage du 2e ordre d'un point O de la courbe jacobienne de l'un d'eux, par ∞^3 transformations osculatrices [1, 2]. Une seule d'entre elles,

canonique, est telle que: le réseau de coniques qui l'engendre contient une droite double; elle transforme tous les éléments tangents en O à la jacobienne en éléments ayant le même contact du 3e ordre que pour T; elle transforme tous les éléments inflexionnels tangents en O à la jacobienne, en éléments ayant le même contact du 4e ordre que pour T. Cela permet un repère projectif privilégié, et 11 invariants projectifs, dont 2 s'interprètent simplement.

P. Belgodère (Paris).

Scherk, Peter. On differentiable arcs and curves. V. On a class of mappings of the curves of order $n+1$ in projective n-space into themselves. Ann. of Math. (2) **47**, 786–805 (1946).

L'auteur dans un article précédent [Ann. of Math. (2) **46**, 68–82 (1945); ces Rev. **6**, 183] a étudié les courbes fermées K^{n+1} d'ordre $n+1$ dans le plan projectif réel R_n. En supposant en chaque point sur K^{n+1} l'existence de variétés linéaires osculatrices à droite de dimensions $i=1, 2, \cdots, n-1$, il a pu définir les notions de point régulier, de point singulier de multiplicité m et de multiplicité d'un point d'intersection de la courbe avec une variété linéaire et retrouver maints théorèmes relatifs aux courbes rationnelles (unicursales) p^{n+1} d'ordre algébrique $n+1$. Les démonstrations faisaient appel à la correspondance continue entre un point s sur K^{n+1} et le point $\mathfrak{f}^n_{n-1}(s)$ où l'hyperplan osculateur $L^n_{n-1}(s)$ coupe de nouveau K^{n+1}. Une variété linéaire de dimension p était appelée spéciale si elle rencontrait K^{n+1} $p+2$ fois et si aucune de ses sous-variétés linéaires de dimension $p-1$ ne rencontrait la courbe $p+1$ fois. Le nombre des variétés linéaires spéciales passant par exactement j différents points de K^{n+1} et contenant la variété osculatrice de dimension p_1 en l'un d'eux, la variété osculatrice de dimension p_2 en un autre, $\cdots$ était désigné par $N^n_{p_1,\cdots,p_r}$. En particulier, N^n_p représentait le nombre des points singuliers de multiplicité $n-p$.

Dans l'article actuel l'auteur discute la correspondance $\mathfrak{f}^n_m(s)$ entre un point s et l'ensemble des points dont les projections à partir de $L^n_m(s)$ sont singulières ($0 \leqq m < n-1$). Pour un s donné l'ensemble $\mathfrak{f}^n_m(s)$ contient au plus $n-m$ points. Les résultats suivants sont établis. (1) Toute application $\mathfrak{f}^n_m(s)$ peut être complétée en une correspondance continue de type $(1, n-m)$ de K^{n+1} en elle-même. (2) La somme $\sum_0^{n-1}(n-p)N^n_p$ augmentée de deux fois le nombre des images réciproques positives [cf. loc. cit., 3.2] d'un point arbitraire de K^{n+1} dans toutes les applications $\mathfrak{f}^n_0(s)$, $\mathfrak{f}^n_1(s)$, $\cdots$, $\mathfrak{f}^n_{n-1}(s)$ est égale à $n+1$. (3) Les nombres N^n_{ij} pour un n donné ($i+j \leqq n-2$) sont bornés [cf. loc. cit., p. 81]. Ces résultats étendent des théorèmes classiques de géométrie algébrique relatifs aux courbes p^3 et p^4.

C. Pauc.

v. Sz. Nagy, Gyula (Julius). Geometrie endlicher Ordnung. Mat. Fiz. Lapok **48**, 207–242 (1941). (Hungarian. German summary)

Der Verfasser gibt einen Überblick über die wichtigsten Ergebnisse aus der im wesentlichen von C. Juel eingeschlagenen Untersuchungsrichtung. Es handelt sich um die gestaltliche Beschreibung der einfachsten projektiv geschlossenen krummen Gebilde endlicher Ordnung mit gewissen Differenzierbarkeitseigenschaften.

From the author's summary.

Arvesen, Ole Peder. Pohlke's theorem. Norsk Mat. Tidsskr. **23**, 100–108 (1941). (Norwegian)

Proof of Pohlke's theorem by means of analytic geometry.

O. Neugebauer (Providence, R. I.).

Beskin, N. An analogue of Pohlke-Schwarz's theorem in central axonometry. Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 57–72 (1946). (Russian. English summary)

The main theorem states: suppose that a space Desargues configuration D' and a plane Desargues configuration D^0 are given; then it is possible to determine a center of projection and a plane of projection in such a way that D' is projected into a plane Desargues configuration D which is equi-affine to D^0.

From the author's summary.

Wunderlich, Walter. Über den "gefährlichen" Rückwärtseinschnitt. Jber. Deutsch. Math. Verein. **53**, 41–48 (italic) (1943).

Three points in a terrain (or on a map) and a photograph of these points are given. It is assumed that the interior orientation of the photograph is known. The problem of resection consists then in finding the location of the instrument station (the viewpoint from which the photograph was taken). This problem has generally four solutions; it may become indeterminate if two of the solutions coincide. It is shown that this is the case if the instrument station is located on a right circular cylinder whose base is the circumcircle of the given triangle. This has already been stated by S. Finsterwalder [Jber. Deutsch. Math. Verein. **6**, no. 2, 1–42 (1899)]. The author investigates this case in detail and proves incidentally several theorems on the Wallace line. For example, an elementary proof is given for Steiner's theorem [J. Reine Angew. Math. **53**, 231–237 (1857)] which states that the envelope of the Wallace lines is a three-cusped hypocycloid.

E. Lukacs.

Di Pisa, Salvatore. Sulla geometria d'un ellissoide a tre assi poco differente da un ellissoide di rotazione. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 460–463 (1942).

Les points de deux ellipsoïdes coaxiaux, de demi-axes

$$\begin{array}{lll} a, & a, & a(1-e^2)^{\frac{1}{2}}; \\ a(1+\frac{1}{2}\mu), & a(1-\frac{1}{2}\mu), & a(1-e^2)^{\frac{1}{2}}, \end{array}$$

avec μ très petit, sont repérés par leurs coordonnées géographiques respectives (φ, ω) et (l, λ), en sorte que le 1er point, sur le 1er ellipsoïde (de révolution) soit situé sur la normale, au 2e point, au 2e ellipsoïde. Le calcul de $(l-\varphi)$, de $(\lambda-\omega)\cos\varphi$, et de la distance relative h/a des deux points, s'effectue directement et algébriquement à l'aide des coordonnées cartésiennes, les termes négligés étant de l'ordre de μ^3.

P. Belgodère (Paris).

Algebraic Geometry

Sansone, Giovanni. La formula di bisezione della $\wp u$ di Weierstrass, e un teorema sui punti razionali delle cubiche ellittiche a coefficienti razionali. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 124–128 (1941).

The author gives a demonstration of the bisection formula for the function $\wp(u)$ of Weierstrass, namely

$$\wp(\tfrac{1}{2}u) = \wp(u) + \sqrt{\wp(u)-e_1}\sqrt{\wp(u)-e_2} + \sqrt{\wp(u)-e_2}\sqrt{\wp(u)-e_3} + \sqrt{\wp(u)-e_3}\sqrt{\wp(u)-e_1}.$$

He states that this formula has not been explicitly mentioned in any of the handbooks of the theory of elliptic functions. [See, however, among others, Whittaker and Watson, A Course of Modern Analysis, 4th ed., Cambridge University Press, 1927, § 20.33, ex. 5; Tannery and Molk, Éléments de la Théorie des Fonctions Elliptiques, v. 1,

Paris, 1893, p. 197 (XVI$_1$).] By means of this formula the following theorem on the cubic C^3: $y^2=4x^3-g_2x-g_3=4(x-e_1)(x-e_2)(x-e_3)$ with rational e_1, e_2, e_3, $e_1>e_2>e_3$, $e_1+e_2+e_3=0$, is obtained. A necessary and sufficient condition that a point (x, y) of C^3, with rational coordinates, and $x>e_1$, is a tangential point of another point (X, Y) of C^3, with rational coordinates, is that $x-e_1$, $x-e_2$, $x-e_3$ are the squares of rational numbers. *S. C. van Veen* (Delft).

Apéry, Roger. Sur les branches superlinéaires des courbes algébriques. C. R. Acad. Sci. Paris **222**, 1198–1200 (1946). [MF 16727]

Let O be the origin of a plane curve branch C with proper parametric representation $x=x(t)$, $y=y(t)$, where $x(t)$, $y(t)$ are power series which vanish at $t=0$. The author considers the totality of orders of polynomials $F(x, y)$ on C, the order of $F(x, y)$ on C being defined as the exponent of t in the leading term of the expansion of $F[x(t), y(t)]$. These orders form a semi-modul (demi-module) N, in the sense that, if n_1 and n_2 belong to N, so does n_1+n_2; N is unitary in the sense that it contains every sufficiently large positive integer. The result obtained by the author is that N is also always symmetrical, in the sense that there exists an integer A (positive or negative, but necessarily odd) such that, of any two integers x, y for which $x+y=A$, one belongs to N and the other does not. There exist, however, symmetrical semi-moduls, for example, that defined by 0, 6, 7, 8 with $A=17$, which are not associated with any branch C.

The author concludes by remarking that, for an algebraic curve branch, $A=2P-1$, where P is the number by which the genus of the curve would be increased if the singularities due to the branch were regarded as virtually inexistent, that is, P is the number of conditions imposed by the branch on adjoint curves. *J. G. Semple* (London).

Amodeo, Federico. Alcune applicazioni del carattere della gonalità delle curve algebriche. Rend. Accad. Sci. Fis. Mat. Napoli (4) **12**, 328–344 (1942).

Amodeo, Federico. Caratteri delle diverse serie canoniche delle curve algebriche e serie residue relative. Rend. Accad. Sci. Fis. Mat. Napoli (4) **13**, 78–90 (1945).

Amodeo, F. Sulle curve piane C_6^7 di ordine sette e di genere sei. Boll. Accad. Gioenia Sci. Nat. Catania (3) **17**, 27–38 (1941).

Brusotti, Luigi. Sul numero dei circuiti delle curve algebriche reali di una quadrica reale. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 113–120 (1942).

Campedelli, Luigi. Le curve gobbe del De Jonquières e del Cremona. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 171–191 (1942).

Cherubino, Salvatore. Dimostrazione algoritmica di un teorema di R. Torelli, nel caso $p=2$. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 21–27 (1942).

We discuss the theorem establishing the transcendental conditions for the birational identity of two algebraic curves. *From the author's summary.*

Franchetta, Alfredo. Sulla curva doppia della proiezione di una superficie generale dell' S_4, da un punto generico su un S_3. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 282–288 (1941).

Miglio, Maria. Covarianti proiettivi di un sistema di l curve dell' S_r a due a due incidentisi. Atti Accad. Gioenia Catania (6) **5**, no. III, 10 pp. (1942).

Niculescu, Alex. Rational Ţiţeica curves. Pozitiva **1**, 56–61, 125–131 (1940). (Romanian. French summary)

Dans cette étude nous prouvons qu'il n'y a pas des cubiques gauches rationnelles, ni des quartiques unicursales qui soient des courbes Ţiţeica. *Author's summary.*

Salini, Ugo. Gruppi satelliti e gruppi tangenziali di specie k sulle curve algebriche piane. Atti Accad. Gioenia Catania (6) **4**, no. XVIII, 8 pp. (1940).

Thalberg, Olaf M. Some harmonic properties of algebraic curves of nth order with an $(n-2)$-fold point. Norsk Mat. Tidsskr. **23**, 116–128 (1941). (Norwegian)

Werenskiold, W. A theorem on curves of 3d order, 3d class, with a cusp and a point of inflection. Norsk Mat. Tidsskr. **23**, 109–115 (1941). (Norwegian)

Tognetti, Mario. Determinazione geometrico-funzionale del gruppo delle terne di punti doppi delle curve di un sistema lineare ∞^3, ciascuna dotata di tre punti doppi. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 711–717 (1942).

Ursino, Grazia. Covarianti proiettivi di curve razionali. Atti Accad. Gioenia Catania (6) **4**, no. V, 9 pp. (1940).

Ce travail fait connaître divers covariants projectifs attachés à une courbe rationnelle. S'appuyant sur la possibilité de regarder une courbe rationnelle d'ordre n plongée dans un espace S_{n-d-1} comme la projection à partir d'un S_d d'une courbe normale du même ordre, et eu égard à une propriété de la polarité de Clifford, l'auteur établit tout d'abord le théorème suivant. La donnée d'une courbe rationnelle d'ordre n plongée dans un S_r et douée d'un S_{r+l-n} k-sécant, individualise projectivement un S'_{n-l-1}, lequel rencontre S_{r+l-n} si l est pair et n impair, et ne le rencontre pas si n est pair ou si n et l sont impairs. Elle fait diverses applications de ce théorème, et associe projectivement: un triangle à une quartique à trois points doubles; un plan (ne passant pas par le point double) à une quartique gauche harmonique; quatre plans (dont les positions relatives sont précisées) à une sextique rationnelle de S_4; une congruence ou une surface réglée (dont les ordres et les classes sont indiqués) à la congruence des cordes d'une courbe gauche rationnelle d'ordre n ou à la réglée de ses trisécantes (avec examen particulier du cas où $n=4$). Généralisant ensuite le théorème du début par la considération d'hyperquadriques, l'auteur démontre, en particulier, que la donnée d'une quartique gauche de 2ème espèce individualise une quartique gauche de 1ère espèce. Puis, grâce à l'introduction d'une certaine homographie Ω, elle aboutit à la conclusion suivante concernant les cubiques c à point double: les points sextatiques de c et ceux de $c'=\Omega c$ sont sur une même conique tangente à c et c' en leurs points sextatiques, et la conique tangente aux tangentes au point double de c et à ses trois tangentes stationnaires est aussi tangente aux trois tangentes stationnaires de c'. Le mémoire se termine par une application de ces derniers résultats à la géométrie différentielle des surfaces. *P. Vincensini.*

Berzolari, Luigi. Sulla configurazione determinata da due cubiche sghembe in posizione ottaedrica. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 11, 69–208 (1941).

Deux cubiques gauches octaédriques sont deux cubiques invariantes simultanément par rapport à un groupe octaédrique de collinéations. De nombreuses propriétés ont déjà été signalées par Kohn [Math. Ann. 52, 293–316 (1899)] et Ciani [Rend. Circ. Mat. Palermo 16, 327–345 (1902); Ann. Mat. Pura Appl. (3) 8, 1–37 (1902)]. L'auteur donne un exposé d'ensemble, avec beaucoup de propriétés nouvelles et curieuses, qui comprend quatre parties.

(1) Nombreuses propriétés assez simples: deux cubiques octaédriques ont six tangentes communes et ceci est caractéristique. Elles sont échangées dans l'involution harmonique qui a pour directrices les sécantes communes h_1h_2 de leurs quatre cordes, etc. (2) Étude des groupes G_{24} qui conserve ces cubiques, G_{48} qui permute les deux courbes, et G_{96} obtenu en adjoignant les polarités qui échangent les courbes avec les développables dont elles sont les arêtes. (3) Étude de certaines surfaces du 4e ordre invariantes dans G_{24} et G_{48}. L'une d'elles, sans point multiple, est très remarquable et possède 54 droites. (4) Étude de deux systèmes Σ_β et Σ_ω à un paramètre de cubiques gauches tracées sur deux réglées rationnelles du sixième ordre ayant h_1h_2 pour directrices triples. Les tangentes de toutes ces cubiques sont extraites du même complexe linéaire et forment une congruence du 4e ordre et de 4e classe. Chacune est en position octaédrique avec une cubique du même système et trois cubiques de l'autre système. En général, toute cubique d'un système admet quatre tangentes communes avec toute cubique de l'autre système. *L. Gauthier* (Nancy).

Gherardelli, Giuseppe. Sulle curve sghembe algebriche intersezioni complete di due superficie. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 4, 128–132 (1943).

Gherardelli, Giuseppe. Sulle curve sghembe algebriche intersezioni semplici complete di tre superficie. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 4, 460–462 (1943).

Gherardelli, Giuseppe. Sul sistema jacobiano di un sistema lineare di curve sopra una superficie algebrica. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 3, 702–710 (1942).

On sait que la variété J des jacobiennes des réseaux extraits d'un système linéaire à r dimensions d'une surface F est rationnelle, mais en général non linéaire (au sens des correspondances sans exceptions). L'auteur détermine ici les cas où la jacobienne J est linéaire. Si on prend pour F un modèle projectif dans lequel le système linéaire est celui des sections hyperplanes, on trouve pour F dans l'espace S_n quelconque (1) les surfaces réglées à sections hyperplanes rationnelles normales; (2) les développables ayant pour arêtes de rebroussement les courbes rationnelles normales; (3) les surfaces réglées de genre arbitraire dont les génératrices appartiennent à la congruence des droites qui, sur un S_1-cône rationnel normal, coupent le sommet D en des points en correspondance homographique avec les plans qui les joignent à D. Il faut ajouter (1) dans S_3 la surface de Veronese, (2) dans S_4 les cônes de genre arbitraire dont la section hyperplane appartient à un complexe linéaire, (3) toutes les surfaces de S_3. *L. Gauthier* (Nancy).

Gherardelli, Giuseppe. Sul modello minimo della varietà degli elementi differenziali del 2° ordine del piano projettivo. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 821–828 (1941).

If V is the three-dimensional manifold of all first order differential elements E_1 of a plane (each element E_1 being defined by a point x and a line u which are incident), Severi has shown [Ann. Mat. Pura Appl. (4) 19, 153–242 (1940), in particular, p. 227; these Rev. 7, 476] that a base for surfaces on V consists of two surfaces X, U corresponding to conditions that an E_1 should have its line u passing through a fixed point x, or its point x lying on a fixed line u, respectively; and the minimum normal model of V is the projective model of $|X+U|$. The second order differential elements E_2 of the plane constitute a fourfold W; and it is now shown that a base for threefolds on W is given by manifolds P, R, analogous to X, U above, together with one or other of two isolated threefolds C, F generated by cuspidal and inflexional E_2, respectively. Also there subsists an equivalence relation $C+3P\equiv F+3R$. By calculating all the intersection numbers, it is found that the minimum normal model of W is the projective model of the system $|C+4P+R|$, and this is nonsingular and of order 330 in [69]. This model is actually constructed by use of coordinates for E_2, invented by Study and Engel; its properties are studied, with reference also to the ordinary second order differential equations which correspond to threefolds of the system $|C+4P+R|$. *J. G. Semple* (London).

Severi, Francesco. Sul limite dell'intersezione di due curve variabili sopra una superficie, le quali tendano ad avere una parte comune. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 3, 410–414 (1942).

A lemma given by the author in the paper reviewed below is contradicted by an elementary example communicated orally by Bompiani. In this note the error is corrected.
O. Zariski (Urbana, Ill.).

Severi, Francesco. La teoria generale dei sistemi continui di curve sopra una superficie algebrica. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 12, 337–430 (1942).

A historico-critical account of the development of the theory of algebraic surfaces in Italy, with principal emphasis on the author's own contributions. The central theme of the exposition is the progress of the problem of completeness of the characteristic series in its well-known alternating phases and controversial aspects. *O. Zariski.*

Severi, Francesco. Sulla irregolarità superficiale d'una varietà algebrica. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 3, 547–555 (1942).

A new geometric proof of the theorem that on a V_3 of irregularity q the variable surface of a sufficiently general linear system has also irregularity q. The theorem is extended to V_r's and linear systems of V_{r-1}'s on the V_r. The proof makes use of the so-called Abelian series on an algebraic curve, hence essentially of the Picard variety of the V_r.
O. Zariski (Urbana, Ill.).

Morin, Ugo. Sulle serie intersezioni complete sopra una superficie algebrica. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 289–293 (1941).

Sur une surface algébrique irréductible F sur laquelle sont donnés deux systèmes linéaires $|c_1|$ et $|c_2|$ de courbes, l'intersection complète des deux systèmes est l'ensemble des

groupes G de points constituant l'intersection d'une courbe $|c_1|$ quelconque par une courbe $|c_3|$ quelconque. A cet égard les surfaces rationnelles jouissent de la propriété que la somme de deux séries intersections complètes est totalement contenue dans une série intersection complète, et F. Severi a montré que si, pour une surface algébrique F, la propriété précedente est vérifiée, un groupe de points G de F appartient à une série intersection complète déterminée et inversement. L'auteur recherche la classe des surfaces jouissant des propriétés indiquées. Il montre que cette classe est celle des surfaces réglées et des surfaces qui peuvent leur être biunivoquement rapportées. Les surfaces autres que les précédentes peuvent être caractérisées par le fait que, pour tout système linéaire de courbes $|c|$ de l'une d'elles on a [G. Castelnuovo-F. Enriques] $n \leqq 2\pi - 2$, $r \leqq \pi$ (n, degré; π, genre; r, dimension). Partant de ce résultat l'auteur montre, à titre de vérification, que sur toute surface non biunivoquement rapportable à une surface réglée, il existe des séries intersections complètes telles que leur somme ne soit pas totalement contenue dans une série intersection complète. *P. Vincensini* (Besançon).

Morin, Ugo. Generazione proiettiva della varietà che rappresenta le coppie non ordinate di punti d'uno spazio lineare S_r. Rend. Sem. Mat. Univ. Padova 12, 123–129 (1941).

Morin, Ugo. Su sistemi di S_h a due a due incidenti e sulla generazione proiettiva di alcune varietà algebriche. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 101, 183–196 (1942).

Conforto, Fabio. Su un classico teorema di Noether e sulle varietà algebriche transformabili in varietà con infinite quadriche. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 268–281 (1941).

Le théorème de Noether dont il s'agit est celui qui établit la rationnalité d'une surface qui contient un faisceau rationnel de courbes rationnelles [Math. Ann. 3, 161–227 (1871)]. Ce théorème a été généralisé par Enriques au cas d'un faisceau irrationnel, la surface étant alors équivalente à une surface réglée [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 7_2, 281–286, 344–347 (1898)]. L'auteur généralise ces théorèmes aux variétés sous la forme suivante. Soit V une variété algébrique de dimension $r+s$ contenant un système S à r paramètres, d'indice ν, d'hyperquadriques à s dimensions généralement non spécialisées. Lorsque $s > 2^r - 2$, la variété V est équivalente à une involution d'ordre ν sur une variété W lieu d'espaces linéaires. Lorsque S est rationnel, W est rationnelle, et V unirationnelle (V est rationnelle si $\nu = 1$, ou si $r = s = 1$, ν quelconque). L'inégalité proposée est seulement suffisante; il serait intéressant d'examiner les valeurs de s voisines de cette limite. Il y aurait également intérêt à rechercher sous quelles conditions une variété contenant un système de sous-variétés rationnelles peut être ramenée au type proposé. *L. Gauthier.*

Zappa, Guido. Sopra una nuova dimostrazione di un teorema di Picard. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 14, 67–72 (1943).

A new proof of Picard's theorem which states that the postulation formula for the double curve of an algebraic surface of order n in S_3 is applicable to surfaces of order not less than $n-4$. The proof is based on a method of degeneration whereby the given surface is forced to degenerate into n planes satisfying specified conditions. The question of whether such a degeneration is always possible is left open. *O. Zariski* (Urbana, Ill.).

Zappa, Guido. Sulle direttrici di grado virtuale minimo d'una rigata algebrica di genere $p > 0$. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 3, 725–733 (1942).

Zappa, Guido. Caratterizzazione delle curve di diramazione delle rigate e spezzamento di queste in sistemi di piani. Rend. Sem. Mat. Univ. Padova 13, 41–56 (1942).

Villa, Mario. Sull'annullarsi, in un punto, della matrice jacobiana di m funzioni in n variabili. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 3, 209–216 (1942).

The osculating $S(r)$ to a curve at a point P is the linear space of minimal dimension containing P and the "derived points" up to order r [Bompiani]. The osculating $S(r)$ to a variety V_h at a point P is the linear space of minimal dimension containing the osculating $S(r)$ to all curves on V_h through P. Pairs of points of two spaces $S_n(x_1, \cdots, x_n)$ and $S_n'(y_1, \cdots, y_n)$ are represented [Segre] by a V_{2n} in $S_{n(n+2)}$, with coordinates $x_1y_1, \cdots, x_1y_n, x_2y_1, \cdots, x_2y_n, \cdots, x_ny_1, \cdots, x_ny_n, x_1, \cdots, x_n, y_1, \cdots, y_n$. The equations (1) $y_i = f_i(x_1, \cdots, x_n)$, $i = 1, \cdots, n$, determine a V_n in the V_{2n}.

Theorem: A necessary and sufficient condition that the Jacobian of (1) vanish at a given point is that the osculating $S(2)$ to the V_n (in $S_{n(n+2)}$) at the corresponding point, and the S_{2n} tangent to the V_{2n}, be contained by a linear space of dimension less than the normal, that is, less than $n(n+5)/2$. The condition is extended to the case of m functions of n variables. Another necessary and sufficient condition: that the S_n tangent to the V_n in $S_{n(n+2)}$ intersect in a straight line the S_n of V_{2n} determined by the point $(y_1, \cdots, y_n)$.

A. B. Brown (Flushing, N. Y.).

Villa, Mario. La configurazione caratteristica di una trasformazione puntuale fra due spazi lineari. I. Proprietà generali. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 4, 137–142 (1943).

Villa, Mario. La configurazione caratteristica di una trasformazione puntuale fra due spazi lineari. II. Applicazioni. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 4, 217–221 (1943).

The characteristic configuration of a point transformation T at a pair (O, O') of corresponding points at which T is regular determines the nature of T in the first and second neighbourhoods of O and O'. In the first paper the author defines this configuration and enumerates (without proof) some of its properties. The configuration consists (i) of the projectivity ω induced by T between the first neighbourhoods of O and O', (ii) of a finite number of corresponding lines (inflexional directions) r_i, r_i' through O, O' such that a curve having an inflexion at O with tangent r_i is transformed into a curve having an inflexion at O' with tangent r_i', (iii) a projectivity between each of these lines r_i, r_i' in which to the three infinitely near points of the inflexion at O correspond the three infinitely near points of the inflexion at O'. The number of inflexional directions, for a transformation between two S_r's, is 3 if $r = 2$ and $7 \cdot 3^{r-3}$ if $r > 2$. The author states conditions sufficient to determine a possible characteristic configuration.

In the second paper the author discusses applications to metrical invariants of point transformations and to the approximation to such transformations by means of Cremona transformations. *J. A. Todd.*

Villa, Mario. Sull'approssimazione delle trasformazioni puntuali fra due spazi mediante trasformazioni cremoniane. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 216–230 (1942).

Godeaux, Lucien. Remarque sur le contact des surfaces. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 391–396 (1946).

Lorsque deux surfaces algébriques F_1F_2 d'ordres n_1, n_2 ont en commun une seule courbe C d'ordre ν sans point singulier, le long de laquelle elles ont un contact d'ordre $p-1$ ($n_1n_2=p\nu$), un raisonnement sur la première polaire montre que l'une au moins des surfaces admet sur C au moins un point singulier A. Si A est double, et si $p=2K+1$ ou $p=2K$, A admet K points doubles infiniment voisins successifs, biplanaires, sauf le dernier qui est conique dans le cas où p est pair. Si A est d'ordre de multiplicité $q>2$, on a $q\leqq p$ et si $q\neq p$ le point admet des points multiples infiniment voisins. Il serait intéressant d'examiner l'aspect arithmétique (valuationnel) de cette question.

L. Gauthier (Nancy).

Martinelli, Enzo. Sulla varietà delle faccette φ-dimensionali di S_r. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **12**, 917–943 (1942).

The author discusses the variety $M^{(r,\nu)}$ of dimension $p+(r-p)(p+1)$, consisting of incident point-S_p elements in an S_r. This variety is clearly birationally equivalent to the product variety of S_r and the Grassmannian of S_{p-1}'s in S_{r-1}. By considering the fundamental elements in this birational correspondence the author proves that the minimum base for varieties of any dimension k on $M^{(r,\nu)}$ consists of the set of varieties denoted by $([a_1, \cdots, a_i, \cdots, a_p]_{k-i})_k$ in which the symbol denotes the aggregate of incident point-S_p elements for which the S_p satisfies a Schubert condition $[a_1, \cdots, a_i, \cdots, a_p]$, where $a_r+\cdots+a_p-\frac{1}{2}p(p+1)=k-i$, and the point lies in the S_i common to S_p and S_{a_i}. He proceeds to construct the variety Z of least order whose points are in unexceptional correspondence with the elements of $M^{(r,\nu)}$. This variety lies in a space of dimension $d-r+p$, where

$$d=(r+1)\binom{r+1}{p+1}-1,$$

and is a partial intersection of this space with the product variety of S_r and the Grassmannian of S_p's in S_r; its order is

$$2^{r-p}\binom{g+p}{p}N,$$

where N is the order of this Grassmannian and g its dimension. *J. A. Todd* (Cambridge, England).

Borrello, Antonio. Nell' S_r i k-complessi ipolineari di rette. Atti Accad. Gioenia Catania (6) **4**, no. IX, 14 pp. (1940). Cf. G. Marletta, same Atti (6) **3**, no. 24 (1939).

Campedelli, Luigi. Sopra i piani doppi con tutti i generi uguali all'unità. Rend. Sem. Mat. Univ. Padova **11**, 1–27 (1940).

Carbonaro, Carmela. Rapporto anarmonico di quattro punti considerati su una ipersuperficie dell' S_r. Boll. Accad. Gioenia Sci. Nat. Catania (3) **18** (1941), 15–17 (1942).

Carbonaro, Carmela. I complessi di regoli, d'ordine uno, dell' S_3 rigato. Atti Accad. Gioenia Catania (6) **5**, no. VIII, 35 pp. (1942).

Dantoni, Giovanni. Determinazione delle superficie con serie di Severi di ordine nullo o negativo. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **14**, 39–49 (1943).

Gandin, Renato. Intorno a due problemi di geometria numerativa ed alla loro interpretazione funzionale. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **100**, 471–478 (1941).

Determinazione del gruppo dei contatti ν-pli degli S_{r-2} aventi un contatto ν-punto ed uno $(r-\nu)$-punto con una C_p^n di S_r. Determinazione del gruppo dei contatti degli S_{r-2} che toccano e $(2r-5)$-secano altrove una C_p^n di S_r.

From the author's introduction.

d'Orgeval, Bernard. Les variétés à trois dimensions de genres I, dont les sections hyperplanes sont surfaces canoniques d'ordre minimum. J. Math. Pures Appl. (9) **25**, 173–178 (1946).

Spampinato, Nicolò. Le riemanniane degli S_r ipercomplessi. Rend. Accad. Sci. Fis. Mat. Napoli (4) **13**, 23–46 (1945).

Spampinato, Nicolò. La varietà W_4^8 dell' S_3 complesso riemanniana del piano biduale. Rend. Accad. Sci. Fis. Mat. Napoli (4) **13**, 128–141 (1945).

Bruno, Carmelo. Le trasformazioni multiple nell' S_r. Atti Accad. Gioenia Catania (6) **5**, no. V, 19 pp. (1942).

Comessatti, Annibale. Sulle plurilinearita tra spazi. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **14**, 23–37 (1943).

Galbura, Giorgio. Sul gruppo caratteristico di una corrispondenza tra varietà algebriche. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **13**, 965–985 (1942).

Differential Geometry

Caldarera, Gaetano. Sulle proiezioni policentriche meridiane. Atti Accad. Gioenia Catania (6) **4**, no. X, 12 pp. (1940).

A series of geometrical projections of the terrestrial sphere are considered. These are projections on a polar plane from a point located on a perpendicular equatorial diameter. The analysis develops several laws of location of the projection point. In particular, a projection from a fixed point is given which includes the stereographic and orthographic projections. Also outlined are two cases which produce, respectively, equally spaced meridians and equally spaced parallels. Tables for these two projections are given. *N. A. Hall.*

Hunger, F. Die ebene konforme und formtreue Abbildung eines geodätischen Strahlenbüschels. Z. Vermessungswesen **72**, 153–160 (1943).

An investigation is made regarding conformal projections of the earth's surface and, in particular, those projections having the property that the geodetic curvature of any

pencil of geodetic lines on the surface is maintained by the projection. It is shown that this property exists for a pencil through any point provided the surface is regarded as a sphere, but that for an ellipsoidal earth it applies only to the pole. *N. A. Hall* (East Hartford, Conn.).

Morelli, Carlo. Formule fondamentali per l'estensione alla quarta approssimazione della trigonometria dei piccoli triangoli curvilinei sopra una superficie qualunque. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **13**, 1035–1061 (1942).

In order to study the trigonometry of a small curvilinear triangle whose sides are not necessarily geodesic arcs on any surface S, the author develops two fundamental formulas to the fourth order of approximation. These are used in the paper reviewed below. In the first section, the author discusses the geodesic polar coordinates (u, v) of a variable point Q on a surface S relative to a fixed point P of S. The Cartesian coordinates (x, y, z) of Q are obtained as power series in (u, v). From this is deduced the value of the Gaussian curvature k of S at Q. Next the author introduces the normal coordinates $x^1 = u \cos v$, $x^2 = u \sin v$. The first problem is to express the first fundamental form, namely, the square of the linear-element, in terms of (x^1, x^2). The coefficients of the various powers of the variables are rational integral combinations of the Gaussian curvature k and its derivatives with respect to x^1 and x^2. Then the Christoffel symbols are expressed in terms of x^1 and x^2. These and their derivatives to the third order are evaluated at the original point P.

By means of the formulas developed, the author develops the power series expansions of the parametric equations $x^1 = x^1(l)$, $x^2 = x^2(l)$ of a curve C which passes through P and Q, where l is the length of arc of C from P to Q, to terms of the fifth degree in l. The coefficients of the various powers of l are rational integral combinations of the Gaussian curvature k and the geodesic curvature γ of C and the derivatives of these with respect to l, all evaluated at the original point P of C. Finally let the curve C be tangent to the curve $x^2 = \text{constant}$ at P. Denote by a the length of the geodesic chord PQ and by θ the angle between the geodesic arc PQ and the curve $x^2 = \text{constant}$. The two fundamental formulas referred to are the power series expansions in l for θ and a to the fourth degree terms. The coefficients are rational integral functions of the Gaussian curvature k and of the geodesic curvature γ of C and their derivatives with respect to l, all evaluated at P.

J. DeCicco (New York, N. Y.).

Morelli, Carlo. Estensione alla quarta approssimazione della trigonometria dei piccoli triangoli curvilinei sopra una superficie qualunque. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 608–618 (1942).

The author develops the trigonometry of a small curvilinear triangle on any surface S of ordinary space to the fourth order of approximation. The sides of such a triangle Δ are not necessarily geodesic arcs. By introducing the related geodesic triangle Δ_g whose vertices are those of Δ and whose sides are geodesic arcs, the author obtains the law of cosines for such a triangle Δ to the fourth order of approximation.

The trigonometry of such a triangle Δ was studied by Darboux [Leçons sur la Théorie Générale des Surfaces, v. 3, Gauthier-Villars, Paris, 1894]. Levi-Civita [Rend. Sem. Mat. Fis. Milano **12**, 1–33 (1938)] found the law of sines to the second order of approximation and Tonolo [ibid. **13**, 35–57 (1939)] extended the law of sines to the third order of approximation.

The law of cosines to the fourth order of approximation is obtained as a consequence of formulas developed by the author in the paper reviewed above. *J. DeCicco.*

Mineo, Corradino. Superficie dotate di ∞^1 geodetiche che sono eliche su cilindri ortogonali a una direzione fissa. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 175–182 (1942).

The idea of extending the notion of geographic coordinates (ϕ, ω) to any nondevelopable surface S is due to Minding [J. Reine Angew. Math. **44**, 66–72 (1852)]. A meridian plane or geographic meridian at a point P of a surface S relative to a fixed direction r is the plane formed by the normal to S at P and the line through P of direction r. The latitude ϕ of P on S is the complement of the angle that the normal to S at P makes with the fixed direction r, and the longitude ω of P on S is the angle that the geographic meridian at P forms with any plane parallel to the fixed direction r. Thus (ϕ, ω) constitute a set of geographic coordinates on S relative to the direction r.

The author gives a complete analytic solution of the problem of determining all those surfaces S on which a family of curves of equal longitude with respect to a given direction r is a family of ∞^1 geodesics. There are three essential types. This problem is equivalent to the determination of all surfaces S for which a family of ∞^1 geodesics (g) are on cylinders which are circumscribed about S and whose generators are perpendicular to the fixed direction r.

This problem may be stated in another form. At any point P of a surface S, the north direction relative to the direction r is defined as the intersection of the geographic meridian with the tangent plane at P, and the east direction is that perpendicular to the north direction and situated in the tangent plane at P. The north and east lines are the integral curves of these two fields of lineal elements defined on S. The preceding problem is equivalent to finding surfaces S for which the lines of equal longitude are isogonals of the east (or north) lines. *J. DeCicco.*

Giambusso, Vincenzo. Linee geografiche su un ellissoide a tre assi. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 441–446 (1942).

Une direction orientée D, et une direction de demi-plan origine issu d'elle, définissent, sur une surface non développable S, des coordonnées géographiques identiques à celles de sa représentation sphérique par parallélisme des normales. Cela définit sur S les lignes d'égale latitude ou longitude, leurs trajectoires orthogonales respectives, et les lignes de pente et de niveau relativement aux plans perpendiculaires à D.

Dans le cas particulier d'un ellipsoïde (utilisable comme géoïde) à 3 axes de longueurs inégales, dont l'un porte D, l'emploi des coordonnées cartésiennes détermine élémentairement les 6 faisceaux de lignes géographiques: les lignes d'égale longitude (contour apparent pour une direction perpendiculaire à D) sont les ellipses de section par les plans méridiens; les lignes d'égale latitude sont des quartiques gauches admettant les mêmes plans de symétrie que l'ellipsoïde; leurs trajectoires orthogonales respectives sont situées sur des cylindres transcendants, d'équations simples. Les "lignes est" sont les ellipses de niveau, et les "lignes nord" (lignes de pente) ont pour projection orthogonale sur le plan équatorial des paraboles, en général transcendantes.

P. Belgodère (Paris).

Colucci, Antonio. Sulla rappresentazione conforme delle superficie a connessione multipla. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 717–721 (1941).

Caccioppoli has shown [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **23**, 287–292 (1936)] that, under certain geometric hypotheses, the Osgood-Carathéodory theorem holds for simply-connected surfaces S bounded by Jordan curves; that is, if the interior of the unit circle Q is mapped conformally on the interior of S, then the mapping can be continued topologically to include the boundaries of Q and S. Pathological surfaces, not satisfying the geometric hypotheses, exist for which the result does not hold.

The method of Caccioppoli extends directly to the case of multiply-connected regions of finite order, but apparently not to the case of regions of infinite connectivity. The author establishes the extension to those surfaces of infinite connectivity for which the bounding curves have only a finite number of accumulation points. *E. F. Beckenbach.*

Pugachev, V. S. Generalisation of the problem of the pursuit curve. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 525–528 (1946). (Russian. English summary)

Point A moves rectilinearly with constant velocity v. Point B moves with constant velocity w so that the tangent to its trajectory at any instant t is directed toward a point C on the trajectory of A, with $AC=f(r)$, where $r=BC$. If A is a moving target, B a pursuit plane with gun barrels fixed parallel to its axis, V the velocity of the shell, and if $f(r)=w/(V+w)$, then C is the point toward which B's axis must be continuously directed if B is to score a hit on A at any instant of fire. An obvious pair of differential equations for the coordinates of B with parameter t are subjected to a transformation T whereby r and t become new dependent variables with z, the negative cosine of B's angle of approach to A's track, as the new parameter. A mapping by T of the integral curves of the (r, z) differential equation in the neighborhood of a singular point reveals the properties of the generalized pursuit curve as $r\to 0$. *T. C. Doyle* (Hanover, N. H.).

Bottema, O. The special Darboux motions in elliptic space. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **53**, 58–65 (1944). (Dutch. German, English and French summaries) [MF 14306]

Darboux has discussed those motions of ordinary three-space in which every point describes a plane curve. These curves are ellipses, except for the trivial case of invariant parallel planes [Koenigs, Leçons de Cinématique, Paris, 1897, pp. 353–360]. Several authors have paid attention to the special case in which there exists an invariant axis. This paper studies the corresponding motions in elliptic three-space with an invariant axis l. Every point P describes a conic section which is the intersection of (a) the plane associated with P with respect to a certain null system and (b) a Clifford surface with l as axis passing through P.

D. J. Struik (Cambridge, Mass.).

Bottema, O. Plane curves with the affine intrinsic equation $k=cp(s)$. Nieuw Arch. Wiskunde **21**, 89–100 (1941). (Dutch) [MF 14296]

The solution of the equation $k=cp(s)$, where k is the affine curvature, s the affine arc length and p the equianharmonic p-function of Weierstrass leads to the solution of the equation $z''+cp(s)z=0$. Comparison with the equation of Lamé, $z''=n(n+1)\{p(s)+a\}z$ (a is constant), allows us to find solutions for the cases $n=1$, $n=2$, corresponding to $c=-2$, $c=-6$, which have been discussed by G. Bol [Abh. Math. Sem. Hamburgischen Univ. **9**, 93–94 (1932); see also Nieuw Arch. Wiskunde **20**, 113–162 (1940), in particular, p. 152; these Rev. **1**, 171]. Curves corresponding to some other values of n are discussed, in particular, the case $n=4$, $c=-20$, which gives the rational cubic curves with three real points of inflection on the line at infinity.

D. J. Struik (Cambridge, Mass.).

Scherrer, W. Über das Hauptnormalenbild einer Raumkurve. Comment. Math. Helv. **19**, 115–133 (1946).

A theorem of Jacobi [Werke, v. 7, p. 39] states that the normal indicatrix of a closed space curve divides the unit sphere into two parts of equal area. The author observes that this theorem is incomplete, since the normal indicatrix need not be simple, and discusses the general case of the theorem. A nonsimple curve is dissected into l simple loops, the number l being called the rotation number of the curve. The sum of the areas positively bounded by these loops is called the area Ω of the curve. The nutation number n of the curve is defined as the number of complete rotations of the Darboux rotation vector relative to the tangent to the curve. The general statement of Jacobi's theorem is given by the formula $\Omega=2(l-n)\pi$. The author might well have observed that, since he allows negative values of the curvature and singular points on the curve, his normal indicatrix is not always the same as that commonly given. This accounts for the apparent discrepancy between the example given in the paper and a theorem by Fenchel [Math. Ann. **101**, 238–252 (1929)] which implies such a construction is impossible. While the paper is substantially correct, the reviewer feels that it is open to the following objections. The proof that the rotation number of a closed curve of class C^3 on the sphere is independent of the initial point [theorem 1] appears unsound, due to the lack of uniqueness in the dissection of the curve. Theorem 2, which states that every closed spherical curve of class C'' has a finite rotation number, is incorrect. A counterexample can readily be given. Indeed, after the first loop is removed, so that the remaining curve is not of class C', the construction given for the succeeding loops may be impossible. In theorems 6 and 11, which have to do with a transformation of the curves, it is obscure in what sense the term "transformation of a curve" is to be understood.

S. B. Jackson (College Park, Md.).

Popa, Ilie. Sur "l'aire" des courbes gauches fermées. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] **1**, 15–23 (1946).

This paper discusses the vector-valued Pfaffian form $(r\times dr)$ (where r is the position vector from a fixed origin, in Euclidean 3-space), the definite integral of this differential form over a closed curve (called the areolar vector of the curve), and the indefinite integral along a curve (which is a new curve, called the areolar curve of the old curve).

H. Federer (Providence, R. I.).

Gheorghiev, Gh. Sur le vecteur aréolaire d'une courbe gauche. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] **1**, 32–45 (1946).

This paper discusses the relation of the areolar vector of a curve [see the preceding review] to the areolar vectors of its six pedal curves. *H. Federer* (Providence, R. I.).

Gołąb, St. Sur la généralisation d'une formule de Lancret concernant l'uniformisation des équations de Frenet. Ann. Soc. Polon. Math. **18**, 129–133 (1945).

Lancret's expression for the Frenet equations of a skew curve in Euclidean three-space states that $dt_i/ds=[l, t_i]$, $i=1, 2, 3$, where t_1, t_2, t_3 are, respectively, the unit tangent, normal and binormal vectors to the curve, $l=k_1t_1+k_2t_2$ (k_1 and k_2 being the curvature and torsion, respectively) and the bracket indicates the vector product of its arguments. In the present paper Lancret's formula is generalized to include the Frenet equations of a curve in a Riemann V_n with positive definite metric. Let $I_{j,j+1}$ be the $(n-2)$-vector $[t_1 \cdots t_{j-1}t_{j+2} \cdots t_n]$ and let $L=\sum_{j=1}^{n-1}k_jI_{j,j+1}$, where k_j are the various curvatures of the curve. Furthermore, if $[v_1 \cdots v_{n-1}]$ is an $(n-1)$-vector, let $[v_1 \cdots v_{n-1}]^*$ be the contravariant vector associated with it by means of the ϵ-tensor. Then the author proves that the Frenet equations in V_n take the form $dt_i/ds=[L, t_i]^*$, $i=1, \cdots, n$, where t_1 is the unit tangent vector and t_i ($i>1$) are the successive unit normal vectors to the curve. *C. B. Allendoerfer.*

Maxia, Angelo. Studio proiettivo differenziale di un elemento cuspidale di specie superiore. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 463–468 (1943).

A curve C is said to have at $O(0, 0)$ a superlinear branch of order m ($m>1$) and class q ($q>0$) if the parametric equations of C may be written in the form $y=t^m$, $x=t^n(a+bt+ct^2+\cdots)$, $n=m+q$. Discussions are made of the curves C for the cases $m<q$, $q=mh+k$ ($k<m$) for $h=1$, $h=2$, $h>2$ and for the cases $m<q$, $m=hq+k$ ($k<q$) for $h=2$, $q=1$, $h>2$, $q=1$ and $q>1$, $k<q$, the corresponding equations of such curves being given. *V. G. Grove* (East Lansing, Mich.).

Marletta, Giuseppe. Rapporto anarmonico di quattro punti considerati su una data curva. Boll. Accad. Gioenia Sci. Nat. Catania (3) **18** (1941), 23–25 (1942).

Let A, B, C, D be four points of a curve Γ in a space S_r. There exists a rational normal curve α of order r passing through B, C, D and having r-point contact with Γ at A. Similarly, by permuting these points, curves β, γ, δ are obtained. It has been shown by P. Calapso that there exists a certain cross ratio a of the four points A, B, C, D considered as belonging to α. Similarly, cross ratios b, c, d are defined. The expression $[A\,B\,C\,D]=a\cdot b\cdot c\cdot d$ is a projective invariant of the four points A, B, C, D considered as belonging to Γ, and is called the cross ratio of A, B, C, D. Several of the properties of the ordinary cross ratio of four points on a line are properties of the expression $[A\,B\,C\,D]$. *V. G. Grove* (East Lansing, Mich.).

Bompiani, Enrico. Sulle superficie con flessi infinitamente vicini. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 497–510 (1941).

A planar point O of order three of a surface S may be defined as a point at which the tangent plane exists and intersects S in a curve with a triple point at O [cf. V. G. Grove, Trans. Amer. Math. Soc. **40**, 155–166 (1936)]. Every other section of S through O has an ordinary inflection at O. The paper under review in effect studies special types of planar points at which the triple point tangents at O coincide. In the terms used by the author, a point O of S is an inflection if the tangent plane contains not only the neighborhood of S at O of the first order, but also that of the second order. Among other things a discussion is made of analytic surfaces and algebraic surfaces having a curve of inflections. *V. G. Grove* (East Lansing, Mich.).

Bompiani, Enrico. Invarianti proiettivi di calotte. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 888–895 (1941).

The author considers in a projective S_3 an element of the second order (calotta) of a surface in the neighborhood of one of its points O. The totality of such surface elements at O having the same tangent plane is a 3-dimensional projective space R_3, the "space of calottes." It contains an invariant plane and quadric cone which can be used as an absolute. With this representation the author develops the geometrical meaning in S_3 of simple projective configurations in R_3 with particular emphasis on the discovery of projective invariants. To cite perhaps the simplest case, two calottes determine in R_3 a line intersecting the absolute in three points and thus determine two independent projective invariants (cross ratios); the calottes of this line regarded in S_3 are the totality of those whose intersections in pairs have the same nodal tangents as the two given ones. The projective generalization of Mehmke's invariant of contact of two surfaces is also brought to light through this representation. It not only reduces to Mehmke's in the Euclidean case but leads to a corresponding theorem in non-Euclidean geometry with a nondegenerate quadric as absolute. The paper closes with an extension to n dimensions, the representation space now having $n(n+1)/2$ dimensions.
J. L. Vanderslice (College Park, Md.).

Buzano, Piero. Sulle calotte del 2° ordine appartenenti a una data striscia. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 485–492 (1942).

A theorem of G. Sannia [C. R. Acad. Sci. Paris **155**, 636–638 (1912)] states: the cross ratio of the normal curvatures of sections of four integral surfaces passing through the same characteristic made by planes parallel to a fixed plane is constant along the characteristic. The author seeks the pure projective theorem which lies behind it using Bompiani's notion of second order "calottes" [see the preceding review]. First he develops analytically the idea of calottes of the second order belonging to a "strip" of planar elements. It is found that those with a fixed center form a pencil of special type: in the Bompiani space of calottes the image line is tangent to the absolute cone. Given four calottes of the strip (with fixed center), the cross ratio of the four image points gives a projective invariant, the desired generalization of the Sannia invariant mentioned above. Their identity in the metric case is demonstrated. A demand that the invariant be independent of center leads to a Riccati equation defining a kind of law of displacement of calottes along the base curve of the strip. When one of the four points is taken on the absolute plane, another at the point of tangency with the absolute cone, the invariant of Mehmke appears. A new property of this invariant is proved. *J. L. Vanderslice* (College Park, Md.).

Dolaptschiew, Bl. Eine Art von Flächenkurven. Zylinderkettenlinien. Mat. Fiz. Lapok **50**, 24–28 (1943). (Hungarian. German summary)

Es werden diejenige Flächenkurven untersucht, welche die Eigenschaft besitzen, dass in jedem Punkt der Kurve die Hauptnormale mit der Flächennormale einen konstanten Winkel ϑ einschliesst. Beispiele solcher Flächenkurven "gleicher Neigung" sind: die Kreise auf der Kugel, die Parallel- und Meridiankurven der Rotationsflächen, die geodetischen Linien ($\vartheta=0$), die Asymptotenlinien ($\vartheta=\pi/2$). Die Differentialgleichung der Flächenkurven gleicher Nei-

gung wird abgeleitet. Bei Rotationsflächen nimmt die Differentialgleichung eine einfachere Form an. Sie wird für Rotationszylinder gelöst und es wird gezeigt, dass bei der Abwicklung des Zylinders in die Ebene die Kurven gleicher Neigung in Geraden und Kettenlinien übergehen.

Sucht man andererseits die Gleichgewichtslage eines schweren, homogenen, unausdehnbaren und biegsamen Fadens, welcher auf einen vollkommen glatten Kreiszylinder mit vertikaler Rotationsachse aufgewickelt ist, so findet man eben die Zylinderkurven gleicher Neigung.

From the author's summary.

Creangă, Ion. Les droites "C" de correspondance. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] **1**, 24–31 (1946).

Soit T une correspondance ponctuelle entre deux surfaces Σ et $\bar{\Sigma}$. Si l'on considère les courbes de Σ issues d'un point O et les courbes homologues issues du point $\bar{O}$ de $\bar{\Sigma}$ correspondant à O, et si l'on associe les plans osculateurs en O et $\bar{O}$ de deux courbes correspondantes quelconques, on obtient une certaine correspondance entre ces plans osculateurs. Cette correspondance est birationnelle du 3ème ordre. Les homologues des plans osculateurs issus de O passant par une droite δ enveloppent un cone de 3ème classe, et, lorsque δ decrit l'étoile de sommet O, E. Bompiani [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **32**, 376–380 (1923); Ann. Mat. Pura Appl. (4) **1**, 259–284 (1924)] a montré que parmi les ∞^2 cônes correspondants l'un dégénère. La droite α issue de O donnant lieu à cette dégénérescence, à laquelle (à une exception près fournie par les correspondances T conservant un système d'asymptotiques au moins) correspond une droite $\bar{\alpha}$ issue de $\bar{O}$, est dite l'axe de la correspondance T relatif à O. L'auteur de la note actuelle établit une propriété duale de celle de Bompiani. Il considère les plans tangents en $\bar{O}$ et O à Σ et $\bar{\Sigma}$, et les points caractéristiques ω et $\bar{\omega}$ de ces plans relatifs à deux courbes homologues quelconques issues de O et $\bar{O}$. Ces points déterminent une correspondance T^* entre les deux plans tangents. Cette correspondance est crémonienne du 3ème ordre. Quand $\bar{\omega}$ decrit une droite dans le plan tangent en $\bar{O}$ à $\bar{\Sigma}$, ω decrit une cubique γ dans le plan tangent en O à Σ. Pour une certaine droite $\bar{d}$ du plan tangent en $\bar{O}$ à $\bar{\Sigma}$ la cubique γ se reduit à trois droites, dont deux sont les tangentes asymptotiques en O, la troisième étant une certaine droite d du plan tangent en O à Σ; d est la droite, dite de correspondance, que l'auteur fait correspondre dualistiquement à l'axe de correspondance de Bompiani.

P. Vincensini (Besançon).

Simonart, Fernand. Sur une transformation généralisée de Ribaucour. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 353–364 (1945).

L'auteur généralise des résultats dus à Ribaucour sur les couples de surfaces nappes de l'enveloppe d'une congruence de sphères. Les congruences formées par les droites joignant un point quelconque de la surface lieu des centres aux deux points de contact de la sphère centrée en ce point avec les deux nappes de son enveloppe, qui peuvent être regardées comme résultant de la réflexion l'une de l'autre sur la surface lieu des centres (dirimante), sont normales, et leurs développables se correspondent et correspondent aux lignes de courbure des deux nappes de l'enveloppe. L'auteur envisage le cas, non plus de la réflexion, mais de la réfraction d'une congruence sur une dirimante donnée avec indice de réfraction ν constant ou variable. Il étudie les cas où la congruence réfractée est normale en même temps que la congruence incidente (ν doit alors être une fonction de la seule distance du point d'incidence à l'une des surfaces orthogonales aux rayons de la congruence incidente), et examine les conditions de correspondance des développables dans les deux congruences incidente et réfractée. L'hypothèse où cette dernière correspondance est réalisée et où en outre ν est constant, conduit à des couples de surfaces en correspondance généralisée de Ribaucour, caractérisées par la relation $R'=-\nu R$, R et R' étant les distances du point d'incidence aux deux surfaces associées, orthogonales l'une aux rayons de la congruence incidente l'autre à ceux de la congruence réfractée. La loi analytique de passage d'une surface à l'une de ses transformées de Ribaucour généralisées est établie par un procédé plus simple que celui employé par Bianchi pour passer d'une surface à l'une de ses transformées de Ribaucour. *P. Vincensini* (Besançon).

Winants, Marcel. Nouvelle contribution à la théorie des lignes de courbure passant par un ombilic. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 33–38 (1946).

É. Picard a rattaché la recherche des lignes de courbure passant par un ombilic d'une surface à l'étude des singularités que présentent les intégrales de certaines équations différentielles du premier ordre et du second degré, et a conclu à l'existence d'une, deux ou trois solutions réelles du problème. L'auteur cite des exemples qui semblent mettre en défaut la conclusion précédente, telle la surface $x^4+y^4+z^4=a^4$ qui admet 14 ombilics dont 8 seulement obeissent à la loi d'É. Picard, chacun des six autres livrant passage à 4 lignes de courbure, ou d'autres surfaces admettant des ombilics par lesquels passent des lignes de courbure en nombre quelconque. Il montre ensuite comment, en modifiant légèrement la théorie d'É. Picard, les exemples exceptionnels qu'il signale en deviennent de simples cas particuliers. Il étudie l'indétermination de l'équation de Picard dans le cas d'une surface de révolution, puis d'une surface admettant une symétrie quaternaire ($a^2z=x^4+y^4$) et, s'appuyant sur les résultats obtenus à propos de ce dernier exemple, étend, dans le sens indiqué, la théorie d'É. Picard au cas d'une surface analytique. *P. Vincensini* (Besançon).

Lalan, Victor. Courbes isothermes sur une surface. Surfaces isothermiques. C. R. Acad. Sci. Paris **223**, 707–709 (1946).

Let ω^1 and ω^2 be two linear differential forms and let $\omega=a\omega^1+b\omega^2$, where a and b are functions of the variables entering into ω^1 and ω^2. Then the general expression is found for a form Ω such that $d\omega=[\omega\Omega]$. The assumption that Ω is an exact differential $d\lambda$ has the consequences: (a) the general integral of $\omega=0$ can be written $U+V=$constant; (b) e^λ is an integrating factor for ω so that $e^\lambda\omega=d\rho$; $\omega=0$ is integrable by quadratures; (d) ρ is of the form $U+V$.

These results are applied to a surface for which ω^1 and ω^2 are the divisors of the corresponding ds^2. It is proved that for the curves $a\omega^1+b\omega^2=0$ to be isotherms it is necessary and sufficient that Ω is an exact differential. Application is also made to surfaces where ω^1 and ω^2 are the pseudo-arcs of the minimal lines. In addition to classical results, the properties of isothermic curves on an isothermic surface are proved. *C. B. Allendoerfer* (Haverford, Pa.).

Lalan, V. Invariants géodésiques d'une courbe minima tracée sur une surface. C. R. Acad. Sci. Paris **223**, 569–570 (1946).

The geodesic trihedral of a minimal curve on a surface is defined to have its first unit vector tangent to the curve, its

second unit vector normal to the surface and its third unit vector normal to the first two. This trihedral is related to the intrinsic trihedral of the curve by a parabolic rotation of amplitude p. This invariant is the analogue of the angle between the normal to the surface and the principal normal to the curve in the case of ordinary curves. It is shown that p can also be interpreted as the geodesic pseudo-curvature of the curve, and the mean curvature of the surface as its geodesic pseudo-torsion. Applications are given to minimal curves lying on minimal surfaces.

C. B. Allendoerfer (Haverford, Pa.).

Zito, Ciro. Sulle reti piane a invariante uguali. Boll. Accad. Gioenia Sci. Nat. Catania (3) **18** (1941), 18–22 (1942).

A determination of all plane nets having equal point invariants, one of whose component families of curves consists of straight lines. *V. G. Grove* (East Lansing, Mich.).

Vincensini, Paul. Sur quelques types spéciaux de réseaux et de congruences conjugués. Ann. Sci. École Norm. Sup. (3) **62**, 269–300 (1945).

The first of the three parts of this study is devoted to the relations between congruences of Ribaucour and conjugate nets with equal point invariants, a congruence of Ribaucour being a congruence whose developables intersect its mean surface in a conjugate net with equal point invariants. It is shown that, given a conjugate net (R) with equal point invariants, a congruence of Ribaucour may be constructed as follows. Construct a conjugate net (R') parallel to (R) (ordinarily (R') is unique up to a homothetic transformation), corresponding points of (R) and (R') being M and M'. Select an arbitrary fixed point I and construct the lines δ through M parallel to IM'. The lines δ generate a congruence of Ribaucour with the sustaining surface of (R) as its mean surface. The net (R') also has equal point invariants. As an application it is shown that, given an arbitrary isothermally orthogonal net (R), and (R') its Christoffel transform, joining the points of (R') to a fixed point I, the lines δ drawn parallel to these through the corresponding points of (R) form the most general congruence of Ribaucour, the sustaining surface of (R) being its mean surface.

The second part applies the results of the first to the study of congruences (H). The most general congruence (H) may be defined as follows. Assume given in a plane π any inversely conformal map, the point M' corresponding to M; the lines through M parallel to IM', where I is any fixed point not in π, generate the most general congruence (H). The relations between harmonic surfaces, that is, surfaces $z=f(x, y)$, where $f(x, y)$ is a harmonic function, and congruences (H) are discussed. The congruences (H) are special congruences of Ribaucour, the mean surfaces being the planes π in the definition. The congruences (H) are shown to be W-congruences.

The third part is devoted mainly to the study of displacements in two parameters. In particular, a study is made of pairs of congruences of lines intersecting at a constant angle α such that the developables of the congruences correspond. A typical result may be stated as follows. Let (R) be a conjugate net with equal point invariants and let (G_1) and (G_2) be two congruences of Ribaucour having (R) as the conjugate net on their common mean surface. Let I_1, I_2 be the fixed points by which (G_1) and (G_2) are defined by means of the parallel net (R') (parallel to (R)) with equal point invariants. If the corresponding lines of (G_1) and (G_2) intersect at a constant angle α then the sustaining surface of (R') is a sphere with diameter I_1I_2 if $\alpha=\pi/2$, or a torus generated by a circle through I_1I_2 if $\alpha\neq\pi/2$. A discussion of the case α not constant is also given and studies of special types of associated congruences and conjugate nets are made. Finally, it is shown, as a by-product of the study, that the minimal (right) conoid may be mapped on an arbitrary surface of revolution with orthogonality of linear elements. *V. G. Grove* (East Lansing, Mich.).

Schneidt, Max. Über eine spezielle Mannigfaltigkeit von W-Strahlensystemen. Deutsche Math. **7**, 75–78 (1942).

All W-congruences are found whose focal surfaces touch cylinders along the curves corresponding to developables of the congruences. *V. G. Grove* (East Lansing, Mich.).

Su, Buchin. General projective theory of curves in space of five dimensions. Acad. Sinica Science Record **1**, 268–274 (1945).

The author utilizes the concept of a representable singularity of order m for a plane curve which was introduced by him [J. Chinese Math. Soc. **2**, 139–151 (1940); these Rev. **2**, 299] in order to simplify the reduction to canonical form of power series expansions for a curve in five-dimensional projective space. He introduces five invariant differential forms and shows that, if the first is not zero, then these forms determine a curve uniquely except for a collineation.

J. E. Wilkins, Jr. (Buffalo, N. Y.).

Su, Buchin. A theorem on surfaces. Acad. Sinica Science Record **1**, 277–282 (1945).

The following theorem, due to the author, is proved in a direct manner. If the quadric of Lie associated with a generic point of a surface always intersects a fixed quadric in a quadrilateral, then the asymptotic curves of the surface must necessarily belong to linear complexes. *P. O. Bell.*

Chang, Su-Cheng. A generalization of quadrics of Moutard. Acad. Sinica Science Record **1**, 337–340 (1945).

The following theorem is established and a generalization of the quadrics of Moutard is thereby determined. Let $p_1, \cdots, p_r$ $(1\leqq r\leqq n-2)$ be r given planes passing through an ordinary point O of a given surface and let p be a plane turning around a nonasymptotic tangent at O of the surface. If an algebraic curve C_n of order n has at O an $(n+3)$-pointic contact with the section of the surface made by p, O for an $(n-1)$-ple point and its $n-2$ tangents at O lying in a preassigned manner in the planes $p_1, \cdots, p_r$, then the locus of C_n is an algebraic surface of order n. *P. O. Bell.*

Hsiung, Chuan-Chih. Some invariants of certain pairs of hypersurfaces. Acad. Sinica Science Record **1**, 332–336 (1945).

Cet article comporte l'énonce de résultats avec seulement quelques indications sur les méthodes suivies. Soient, dans un espace S_n à n dimensions, deux hypersurfaces V_{n-1}, V'_{n-1} ayant un hyperplan tangent commun t_{n-1} en deux points ordinaires O, O'. L'auteur déclare qu'un seul invariant projectif I peut être déterminé, comme dans l'espace ordinaire, par les voisinages du second ordre des hypersurfaces aux points O et O' et il caractérise cet invariant projectivement et métriquement.

Dans une seconde partie, l'auteur considère deux hypersurfaces V_{n-1}, V'_{n-1} telles que les hyperplans tangents t_{n-1}, t'_{n-1} aux points ordinaires O, O' soient distincts mais que

chacun d'eux contienne OO'. Il déclare que "deux invariants projectifs I, J peuvent être déterminés par les voisinages du second ordre des hypersurfaces aux points O, O'; l'un d'eux, J, par exemple, se réduit à 1 quand l'espace S_n est à trois dimensions." L'invariant I généralise l'invariant de Buzano [Boll. Un. Mat. Ital. (2) 14, 93–98 (1935)] dans l'espace à trois dimensions. L'auteur termine en caractérisant métriquement les deux invariants I et J. *M. Decuyper.*

Bochner, S. Vector fields and Ricci curvature. Bull. Amer. Math. Soc. 52, 776–797 (1946).

The author proves theorems concerning the nonexistence of certain types of vector fields and groups of transformations in a Riemannian manifold M or in a complex Hermitian manifold H whose mean (Ricci) curvature has constant sign. Of particular interest are the results obtained for compact manifolds. For example, (1) a compact M with positive mean curvature has no vector field whose divergence and curl both vanish, (2) a compact M with negative mean curvature has no continuous group of isometries, (3) a compact H with negative mean curvature has no continuous group of analytic homeomorphisms, (4) a compact H with negative (positive) mean curvature has no analytic contravariant (covariant) tensor field, (5) if a compact H with positive mean curvature is covered by a finite number of neighborhoods, if a meromorphic functional element is defined in each neighborhood and if the difference of any two meromorphic elements is holomorphic wherever the elements overlap, then there exists one meromorphic function on H which differs by a holomorphic function from each meromorphic element given. *S. B. Myers.*

Schwartz, Abraham. On higher normal spaces for V_m in S_n. Amer. J. Math. 68, 660–666 (1946).

Two theorems are proved concerning the second normal space. In the first theorem the author determines an upper bound for its dimension number which is an improvement on a corresponding bound given by Schouten and Struik [Einführung in die neueren Methoden der Differentialgeometrie, v. 2, 2d ed., Noordhoff, Groningen, 1938, p. 116]. The second theorem is chiefly concerned with the possibility of predicting the vanishing of the second normal space. Allendoerfer has shown [Amer. J. Math. 61, 633–644 (1939); these Rev. 1, 28] that the second normal space vanishes at a particular point if the "type" is at least 3. The theorem proved in this paper predicts the vanishing of the second normal space in cases where Allendoerfer's theorem does not. *E. T. Davies* (Southampton).

Bortolotti, Enea. Coordinate normali ed "estensioni" nella geometria degli spazi a connessione lineare. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 106–116 (1940).

Etant donné un espace A_n à connexion affine, symétrique ou asymétrique, l'auteur se préoccupe d'obtenir des interprétations géométriques des notions liées à l'utilisation de coordonnées normales: extensions d'un tenseur, tenseurs normaux de A_n. Il y parvient par l'introduction systématique de repères non-holonomes dits normaux. Un repère non-holonome normal de pôle P_0 est constitué par une distribution, dans le voisinage de P_0, de repères cartésiens locaux satisfaisant à la condition suivante: les vecteurs fondamentaux du repère en un point P quelconque se déduisent de ceux d'un même repère initial donné en P_0, par transport par équipollenie le long de la géodésique joignant P_0 à P. A un système de coordonnées normales y^r de pôle P_0, se trouvera ainsi associé le repère normal non-holonome défini par les exemples:

$$X^r_i = \delta^r_i - (\Gamma^r_{st})_0 \delta^s_i y^t - \tfrac{1}{3}(\nabla_p \Gamma^{(r)}_{st})_0 \delta^s_i y^t y^p - \cdots,$$

$$X^i_r = \delta^i_r + (\Gamma^s_{rt})_0 \delta^i_s y^t + \tfrac{1}{3}(\nabla_p \Gamma^s_{(rt)})_0 \delta^i_s y^t y^p + \cdots.$$

Si E_n désigne l'espace affine tangent, que l'on rapportera au repère précédent, à tout champ de tenseurs E_n, on peut faire correspondre, par un procédé géométrique explicite, des tenseurs-extensions (non-holonomes) successives de celui-ci. En particulier la connexion de A_n est définie, par rapport à un repère normal non-holonome, par des paramètres dont les valeurs en P_0 engendrent dans E_n un tenseur. Celui-ci exprime la différence entre la connexion de A_n et la connexion

$$\Lambda^r_{st} = X^r_i \frac{\partial X^i_s}{\partial u^t} \tag{1}$$

associée par Weitzenböck à de tels ennuples. On peut appliquer à ce tenseur le procédé d'extension et en déduire une suite de tenseurs normaux (non-holonomes). A cette suite, il convient d'adjoindre la suite de tenseurs normaux déduits par extension de l'indicateur de non-holonomie. Ceux-ci font intervenir la connexion symétrique associée à la connexion (1). Ces résultats s'étendent naturellement aux variétés non-holonomes. *A. Lichnerowicz* (Strasbourg).

Bortolotti, Enea. Varietà a connessione proiettiva e loro immagini tangenziali. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 251–267 (1941).

Soit X_m une variété holonome plongée dans un espace P_n à connexion projective au sens de Cartan. Si Γ désigne une courbe issue d'un point O de X_m et tracée sur X_m, on représentera par $I(\Gamma)$ son image tangentielle sur l'espace projectif S_n tangent en O à P_n; $I(X_m)$ sera l'image tangentielle de X_m relative à O, c'est-à-dire l'ensemble de toutes les $I(\Gamma)$. L'auteur établit que $I(X_m)$ peut être approchée au 4° ordre par une variété non-holonome de S_n. On notera que les éléments du 2° ordre de $I(\Gamma)$ en O appartiennent à une même calotte holonome à m dimensions de S_n, mais qu'il n'en est plus de même en général pour les éléments du 3° ordre. Ces résultats établissent une relation étroite entre les travaux de Cartan relatifs par exemple à une surface holonome plongée dans un P_3 (tangentes conjuguées, tangentes de Darboux) et ceux analogues relatifs aux surfaces non-holonomes plongées dans un S_3 [Maxia, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 21, 248–253 (1935)]. *A. Lichnerowicz* (Strasbourg).

Varga, Otto. Über eine Charakterisierung der Riemannschen Räume konstanter Krümmung. Mat. Fiz. Lapok 50, 34–39 (1943). (Hungarian. German summary)

Bekanntlich ist der Riemannsche Krümmungsskalar für eine Flächenstellung definiert und daher im Falle, dass der Raum von höherer als zweiter Dimension ist, nicht nur eine Ortsfunktion, sondern auch eine Funktion der Flächenstellung. F. Schur hat bewiesen, dass aus der Unabhängigkeit der Krümmung von der Stellung auch die Unabhängigkeit vom Orte folgt. In der vorliegenden Note wird gezeigt, dass der Raum auch dann von konstanter Krümmung ist, wenn der Krümmungsskalar vom Orte unabhängig ist.

Author's summary.

Varga, O. Linienelementräume deren Zusammenhang durch eine beliebige Transformationsgruppe bestimmt ist. Acta Univ. Szeged. Sect. Sci. Math. 11, 55–62 (1946).

Un groupe de Lie arbitraire G étant donné, l'auteur explicite, selon la méthode du repère mobile de É. Cartan, la construction des variétés dont l'élément générateur est un élément linéaire et qui admettent une connexion du groupe G. Les espaces de Finsler, les espaces variationnels généralisés de Lichnerowicz constituent les exemples les plus connus de telles variétés, le groupe G étant alors le groupe euclidien. On trouvera en particulier dans cette note une théorie de la courbure de ces variétés dans le cas d'un groupe quelconque. *A. Lichnerowicz* (Strasbourg).

Varga, O. Zur Differentialgeometrie der Hyperflächen in Finslerschen Räumen. Deutsche Math. 6, 192–212 (1941).

Papers on the theory of hypersurfaces imbedded in a Finsler space have been written from two points of view, according as the line "element of support" is tangential or normal to the hypersurface. In this paper it is assumed that the line element is tangential. The first part of the paper is mainly expository. In the second part the author proves the theorem that among the infinite number of representations of an alternating form of the pth degree,

$$F\equiv\sum_{i_1<\cdots<i_p}^{1,\cdots,n-1} V_{i_1 i_2\cdots i_p}[\omega^{i_1}\cdots\omega^{i_p}],$$

there is one and only one whose coefficients satisfy the relation $l^k V_{k i_2\cdots i_p}=0$, where ω^i is the absolute differential of l^i. He applies this theorem to the 4-index curvature tensors of the hypersurface and ends with a generalization of the "theorema egregium" of Gauss. *E. T. Davies.*

Varga, O. Bestimmung des invarianten Differentials in Finsler'schen Räumen. Mat. Fiz. Lapok 48, 423–435 (1941). (Hungarian. German summary)

Der zum Aufbau der Finsler'schen Geometrie wesentliche Begriff des Invarianten-Differentials wird von É. Cartan folgendermassen hergeleitet. Ist ξ^i ein Vektor im Linienelement $(x, \dot{x})$, dann soll das invariante Differential $D\xi^i$ das man erhält, wenn man zum Nachbarelement $(x+dx, \dot{x}+d\dot{x})$ übergeht, linear in $d\xi^i$, ξ^i, $d\dot{x}^i$ und dx^i sein, also die Gestalt

$$(a)\qquad D\xi^i=d\xi^i+C^i_{kl}(x,\dot{x})\xi^k d\dot{x}^l+\Gamma^i_{kl}(x,\dot{x})\xi^k dx^l$$

besitzen. Die Übertragungsparameter C^i_{kl} und Γ^i_{kl} sollen sich eindeutig aus der Grundfunktion $L(x, dx)$ des Finsler'schen Raumes bestimmen lassen. Um dies zu bewirken, stellt É. Cartan neben der selbstverständlichen Forderung, dass die Übertragung metrisch sein soll, noch vier weitere ziemlich verwickelte Forderungen.

In der vorliegenden Arbeit wird das in Frage stehende invariante Differential auf eine andere Weise hergeleitet, die vielleicht übersichtlicher und mehr geometrisch ist. Der Grundgedanke ist folgender. Längs einer stetig differenzierbaren Folge von Linienelementen

$$(b)\qquad x^i=x^i(\tau),\quad \dot{x}^i=\dot{x}^i(\tau),\qquad \tau_0\leqq\tau\leqq\tau_1,$$

wird die Finsler'sche Massbestimmung durch eine oskulierende Riemann'sche approximiert. Das für diese Riemann'sche Massbestimmung wohlbekannte invariante Differential wird nun als invariantes Differential längs der Linienelementfolge (b) erklärt. Das aus dieser Festsetzung heraus bestimmte invariante Differential erweist sich mit dem Cartan'schen (a) identisch.

From the author's summary.

Raševskiĭ, P. Polymetric geometry. Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] 5, 21–147 (1941). (Russian) [MF 15609]

The aim is to construct a generalized geometrical system which preserves as much as possible (only locally, of course) the metric duality typical of the elliptic plane; that means that in the new geometry there should exist analogues of points with distances and straight lines with angles and that these two things should have equal rights. The purpose, then, is to deal with two metrics which have equal rights; such a geometry would be called bimetric. By way of introduction the author considers a Euclidean plane in which the line element is chosen as the fundamental object, and contiguity ($dy=zdx$) as the fundamental relation; properties to be considered are those left invariant by contact transformations. One invariant conception is that of a "curve" defined as a one-parameter family of line elements such that two neighboring ones are contiguous; points and ordinary curves are noninvariant special cases of this. A metric is introduced by specifying a first order differential form which on "curves" gives ds. There exists an allowable transformation of coordinates which takes null-curves (those on which $ds=0$) into points; in the new "canonical" coordinate system the metric reduces to a Finsler metric; among other things canonical coordinates help to associate with a given metric "its angular metric" which gives infinitesimal angle $d\theta$ between two contiguous elements of a null curve of the given metric; the geodesics of the given metric serve as null curves for its angular metric.

After these preliminaries the author introduces two independent metrics each given by a first order differential form, the two having equal rights; the two metrics determine two differential operators (infinitesimal transformations of the Lie theory). One can also give the system by giving the operators; they determine the metrics; the system possesses six invariants. In terms of this analytical apparatus properties of the metrics are expressed in very compact form. Different relationships between the two metrics are investigated; one may be geodesically conjugate, metrically conjugate, and completely conjugate to the other, etc. The case which occupies most of the author's attention is what he calls a dual system; here each metric is conformal to the angular metric of the other. The ideas of triangle, area and some notions of trigonometry appear in generalized and dualistic form.

After a chapter in which the preceding considerations are treated in terms of the Grassmann-Cartan calculus and in which the Gauss-Bonnet theorem is generalized for the dual system the author passes to a generalization of his theory to three-dimensional space. In trimetric geometry the place of line elements is taken by autopolar tetrahedra (the space for convenience is assumed to be elliptic) which constitute the fundamental objects; the system is given by six Pfaffians which satisfy certain algebraic and integrability conditions permitting one to introduce analogues of points and planes. In terms of these Pfaffians two metrics are introduced which have equal rights; one gives distances between "points," the other between "planes"; there is also an intermediate metric giving angles between directions but straight lines are not introduced. A canonical system of coordinates leads, in analogy with the two-dimensional case, to "arc length" which depends on direction and thus to an analogue of a Finsler metric which the author calls pseudo-Finsler because it differs from the usual Finsler metric in three dimensions.

G. Y. Rainich (Ann Arbor, Mich.).

Maeda, Jusaku. Differential Laguerre geometry of plane curves. Jap. J. Math. **18**, 385–581 (1942). [MF 14973]

If u, v are real numbers, $j^2=0$, then $z=u+jv$ is called a dual number. We write $u=\Re(z)$, $v=\mathfrak{D}(z)$, $\bar{z}=u-jv$. If the equation of a straight line in the Euclidean (x, y)-plane is given in the form $(u^2-1)x+2uy=v$, then $z=u+jv$ may be considered as the coordinate of the directed straight line. The Laguerre group consists of all the direct transformations $z^*=(az+b)/(cz+d)$ and the indirect transformations $z^*=(a\bar{z}+b)/(c\bar{z}+d)$, where a, b, c, d are dual numbers with $\Re(ad-bc)\neq 0$ [cf. Blaschke, Vorlesungen über Differentialgeometrie, v. 3, Springer, Berlin, 1929, pp. 173–175]. Let z be a function of the real variable t. If the straight lines $z(t)$ envelop a curve C, $z=z(t)$ may be considered as the equation of C. The Schwarz derivative $\{z, t\}=(z''/z')'-\frac{1}{2}(z''/z')^2$ is an invariant of C under the direct Laguerre group, and the Laguerre arc length $p=\int|\mathfrak{D}\{z, t\}|^{\frac{1}{2}}dt$ is even invariant under transformations of the parameter. This gives the other invariants of C: $\operatorname{sgn}\mathfrak{D}\{z, t\}$ and the Laguerre curvature $K=2\Re\{z, p\}$.

The author deals with a great variety of topics, mostly in close formal analogy to his paper on the Moebius geometry of plane curves [Jap. J. Math. **18**, 67–260 (1942); these Rev. **7**, 265]. In particular, the analogues to the items (1)–(9) of that review are discussed. Ad (1), (2). If a tangent $z(p)$ of C is given, then the straight line ζ is a covariant of $z(p)$ if and only if the "relative coordinate of ζ with respect to C at $z(p)$" $Z=2z'/(z-\zeta)-z''/z'$ is an invariant [the author uses $Z=2z'/(z-\zeta)+z''/z'$ throughout]. If ζ depends on p, $Z=Z(p)$ will be the relative equation of the curve $\zeta(p)$. Then $Z(p)=0$ defines the satellite curve; K and the curves of constant K are characterised in terms of the osculating circles of the satellite curve. [The reviewer would suggest the discussion of the invariant $(z-\zeta)^2/(z'\zeta')$ of two covariantly transformed curves.] Ad (5). The author studies the infinitely many osculating curves of constant K with contact of order 3 or 4 at a given point of C. [In general, there exists exactly one such osculating curve with contact of order 5. Do analogues to the 6-vertex theorem of affine differential geometry and to Boehmer's theorem on elliptically curved ovals hold?] Ad (9). The analogues of the bicircular quartics are the hypercycles. They are given by an equation $av^2+f(u)v+g(u)=0$, where $a=$constant, and $f(u)$ and $g(u)$ are polynomials of the second and fourth degrees, respectively [cf. Blaschke, Monatsh. Math. Phys. **21**, 3–60 (1910)]. The natural equations of the various types of hypercycles are derived and the osculating hypercycles of a given C are discussed. The last paragraph contains a new proof of the fact that the extremals of $\int_{t_1}^{t_2}|\mathfrak{D}\{z, t\}|^{\frac{1}{2}}dt$ are the curves of constant K [cf. Liebmann, J. Reine Angew. Math. **154**, 15–19 (1924)]. *P. Scherk.*

Cisotti, Umberto. Immagine geometrica di un tensore isotropo. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 651–653 (1942).

It is well known that any second order isotropic tensor in Euclidean space can be represented at each point as a spherical surface centered at the point. Taking the expression for the most general isotropic tensor written in Cartesian coordinates the author shows that it also has a spherical surface associated with it, a geometric interpretation which he considers "useful and expressive."

J. L. Vanderslice (College Park, Md.).

Cisotti, Umberto. Campi tensoriali potenziali. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 129–141 (1940).

This paper contains formal results in a new symbolism. The space considered is Euclidean 3-space, and rectangular Cartesians y_i are used. The author states that the generalization to Riemannian space is immediate, but this is not evident to the reviewer, since the noncommutativity of covariant differentiation may give rise to complications. Generally speaking, a tensor field is called a potential field if it can be obtained by differentiation from a scalar field. In particular, if $a_{i_1\dots i_m}=\partial^m f/\partial y_{i_1}\cdots\partial y_{i_m}$, the author writes $\mathbf{a}=\operatorname{grad}_m f$, and calls $\mathbf{a}$ a potential tensor of the first kind. If

$$b_{i_1\dots i_m}=\frac{\partial g}{\partial y_{i_1}}\cdots\frac{\partial g}{\partial y_{i_m}},$$

he writes $\mathbf{b}=\operatorname{Grad}_m g$, and calls $\mathbf{b}$ a potential tensor of the second kind. Divergence of $\mathbf{a}$ or $\mathbf{b}$ is defined by differentiation and contraction. As a sample of results, the following may be quoted: $\operatorname{div}\operatorname{Grad}_2 g=\frac{1}{2}\operatorname{grad}(\Delta_1 g)^2+\Delta_2 g\cdot\operatorname{grad} g$. The author considers as potential tensors the unitary tensor (or Kronecker delta), quadruple and m-uple isotropic tensors, and the inertia tensor. Linear invariants and semi-invariants of potential tensors are discussed. For example, a $2m$-uple potential tensor of the first kind admits a single linear invariant $\Delta_{2m}f$, where f is the potential function and Δ_{2m} the Laplace differential operator applied m times.

J. L. Synge (Pittsburgh, Pa.).

Cisotti, Umberto. Sistemi continui conservativi. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 294–301 (1941).

The stress tensor Φ in a continuous medium is said to be conservative if either (a) a function Ψ exists such that $\Phi=\operatorname{grad}_2\Psi$, or (b) a harmonic function χ exists such that $\Phi=\operatorname{Grad}_2\chi$ [see the preceding review]. The former definition is applied, in particular, to a gravitating body, where the function Ψ is taken to be

$$\Psi(P)=(f/8\pi)\int_S\mu(0)|P-0|dS;$$

here f is the gravitational constant, μ the density, dS the volume element at 0, and the integral extends through the body. A state of stress Φ is then calculated from $\Phi=\operatorname{grad}_2\Psi$, and it is shown that this state of stress is in equilibrium with the body force due to gravity. The case where $\Phi=\operatorname{Grad}_2\chi$ is shown to be connected with Maxwellian stresses. *J. L. Synge* (Pittsburgh, Pa.).

Cisotti, Umberto. Invarianti quadratici dei tensori. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 511–516 (1941).

The author extends his study of linear invariants [same Rend. (7) **1**, 337–341 (1940); these Rev. **1**, 325] to discuss the quadratic invariants formed from a tensor $T_{i_1\dots i_m}$ in a Euclidean 3-space with rectangular Cartesian coordinates. These are of the form $J_r=T_{i_1\dots i_m}T_{k_1\dots k_m}\Delta^{(r)}_{i_1\dots k_m}$, where $\Delta^{(r)}_{i_1\dots k_m}$ are the distinct expressions obtained by permutation of the subscripts in the product of Kronecker deltas $\delta_{i_1k_1}\delta_{i_2k_2}\cdots\delta_{i_mk_m}$. For a vector ($m=1$), there is just one quadratic invariant, the square of the magnitude of the vector. For a tensor of the second order ($m=2$), there are three quadratic invariants; these may be written S^2+E^2, S^2-E^2 and $(T_{ii})^2$, where S and E are, respectively, the symmetric and skew-symmetric parts into which the given

tensor may be decomposed. For a tensor of the third order ($m=3$), there are fifteen distinct quadratic invariants; if the tensor is symmetric, these reduce to two, $T_{ab}T_{ab}$ and $T_{ab}T_{ba}$. The case of potential tensors [see the second preceding review] is considered. *J. L. Synge.*

Cisotti, Umberto. Campi tensoriali potenziali di rango superiore. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 586–591 (1941).

Extending the ideas of an earlier paper [see the third preceding review], the author considers a tensor **T** obtained by taking partial derivatives of another tensor **P**, in a Euclidean 3-space with rectangular Cartesian coordinates. If **P** is of order p, then **T** is said to be a potential tensor of rank $p+1$. In particular, two such potential tensors are considered, both derived from a vector v_m. The first is

$$V_{ik}=\frac{\partial v_m}{\partial y_i}\frac{\partial v_m}{\partial y_k};$$

it occurs in the theory of finite strain. The second may be written

$$W_{ikk}=\epsilon_{mnp}\frac{\partial v_m}{\partial y_i}\frac{\partial v_n}{\partial y_k}\frac{\partial v_p}{\partial y_k}.$$

The author establishes in detail the tensor character of these expressions for orthogonal transformations; the tensor character in question appears obvious to the reviewer [cf. H. Jeffreys, Cartesian Tensors, Cambridge University Press, 1931]. *J. L. Synge* (Pittsburgh, Pa.).

Cisotti, Umberto. Analisi tensoriale e interpretazioni meccaniche. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **14**, 1–9 (1943).

The paper deals with tensors in a Euclidean 3-space with rectangular Cartesian coordinates. If $T_{i_1\dots i_m}$ is a tensor and n_i a unit vector, then the normal component of **T** with respect to **n** is defined to be the invariant $T_{i_1\dots i_m}n_{i_1}\cdots n_{i_m}$. Following Pastori [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **12**, 499–502 (1930)], the author writes for the general isotropic tensor of order $2m$

$$(*)\qquad C_{i_1\dots i_mk_1\dots k_m}=\sum_{r=1}^{N}A_r\Delta^{(r)}_{i_1\dots k_m},$$

where A_r are arbitrary scalars and the Δ's are as described in the second preceding review; N is the number of distinct Δ's. He shows that the normal component of an isotropic tensor with respect to any direction is $\sum_{r=1}^{N}A_r$. Similarly, for a skew-isotropic tensor, the normal component is zero. Furthermore, the author considers the decomposition of a tensor **T** of even order $2m$ into the sum of an isotropic tensor **C** and another tensor **U** orthogonal to **C**, so that the following conditions are to be satisfied:

$$(**)\qquad \mathbf{T}=\mathbf{C}+\mathbf{U},\quad C_{i_1\dots k_m}U_{i_1\dots k_m}=0.$$

The decomposition involves the linear invariants of T,

$$I_s=T_{i_1\dots k_m}\Delta^{(s)}_{i_1\dots k_m}.$$

A formula is given which determines the scalars A_r in $(*)$ linearly in terms of I_s; then, if **U** is determined to satisfy the first of $(**)$, the second is automatically satisfied. It turns out that all the linear invariants of **U** vanish. Also, the choice of the scalars A_r giving this orthogonal decomposition is that choice which minimizes the magnitude of the residue $\mathbf{T}-\mathbf{C}$. The orthogonal decomposition of a tensor of odd order is also discussed; here one of the tensors involved is skew-isotropic. *J. L. Synge* (Pittsburgh, Pa.).

Cisotti, Umberto. Invarianti ed eminvarianti cubici dei tensori. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 8–11 (1943).

In Euclidean 3-space with rectangular Cartesian coordinates, the cubic invariants of a tensor of even order $2m$, $T_{i_1\dots i_mk_1\dots k_m}$, are here defined by the formula

$$(*)\qquad C_r=T_{i_1\dots i_mk_1\dots k_m}\cdot T_{j_1\dots j_mk_1\dots k_m}\cdot T_{p_1\dots p_mq_1\dots q_m}\cdot\Delta^{(r)}_{i_1\dots q_m},$$

where the Δ's are the distinct expressions obtained by permutations of the suffixes in the product

$$\delta_{i_1k_1}\cdots\delta_{i_mk_m}\cdot\delta_{j_1k_1}\cdots\delta_{j_mk_m}\cdot\delta_{p_1q_1}\cdots\delta_{p_mq_m}.$$

Similarly, the cubic semi-invariants of a tensor

$$T_{ikli_1\dots i_{m-1}k_1\dots k_{m-1}}$$

of odd order $2m+1$ are defined by the formula

$$C'_r=T_{ikli_1\dots i_{m-1}k_1\dots k_{m-1}}\cdot T_{jkrj_1\dots j_{m-1}k_1\dots k_{m-1}}\cdot T_{pqsp_1\dots p_{m-1}q_1\dots q_{m-1}}\Delta^{(r)}_{ikl\,jkr\,pqs},$$

where the Δ's are the distinct expressions obtained by permutation of subscripts in a product of three terms of which the first is $\epsilon_{ikl}\delta_{i_1k_1}\cdots\delta_{i_{m-1}k_{m-1}}$, and the other two are obtained from this by changing (ikl) into (jkr) and (pqs), respectively. The formula $(*)$ is used to evaluate explicitly the 15 cubic invariants of T_{ij}; of these only 5 are distinct. *J. L. Synge* (Pittsburgh, Pa.).

Pastori, Maria. Operatori differenziali di ordine superiore negli spazi di Riemann. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 1–14 (1941).

L'auteur étudie systématiquement les opérateurs classiques gradient, divergence, rotationnel appliqués à un tenseur d'ordre quelconque. Il donne un certain nombre de formules portant sur ces opérateurs et leurs itérés, dans le cas d'espaces riemanniens généraux ou à courbure constante ou euclidiens, et indique quelques applications à des problèmes d'élasticité. *A. Lichnerowicz* (Strasbourg).

MECHANICS

Locatelli, P. Principi della meccanica nella scienza delle costruzioni. Rend. Sem. Mat. Fis. Milano **14**, 42–67 (1940).

Jonas, Hans. Ein Schliessungssatz für Gelenkantiparallelogramme und seine räumliche Verallgemeinerung. Deutsche Math. **7**, 152–172 (1943).

The author considers the linkage mechanism of twelve links connected like the edges of a parallelepiped. These links are composed of three sets of four equal links of lengths k_1, k_2, k_3. To each corresponds a skew four-bar linkage in which the opposite links are equal in length and so restrained that the variable angles ω and ω' made by a link with its adjoining links are given by the relation $(\cos\omega\cos\omega'+1)/(\cos\omega+\cos\omega')=D=$constant. The author calls this a Bianchi quadrilateral because it was considered by Bianchi in 1905 and 1908. However, he seems to be unaware of its earlier consideration by G. T. Bennett [Engineering **76**, 777–778 (1903)] and papers and books by Bricard, Dunkerley, Myard, Krames, Delassus, Federhofer and others in which it is called the Bennett linkage. Several of the results of this paper are found in these earlier references which are listed in papers by the reviewer [Trans. Amer. Soc. Mech. Eng. **65**, 649–661 (1943); J. Math. Phys. Mass. Inst. Tech. **25**, 96–110 (1946); these Rev. **6**, 74; **8**, 99].

With each skew quadrilateral is associated a constant C, defined by $C^2=2k_1k_2D-k_1^2-k_2^2$, which vanishes when the quadrilateral degenerates into a parallelogram. For the three quadrilaterals adjoining a vertex of the parallelepiped, the sum of the corresponding values of C vanishes. Other such relations are derived, including relations between opposite sets of trihedral angles. The author considers special cases corresponding to movable Möbius tetrahedra and plane degenerate cases involving Bäcklund transformations.
M. Goldberg (Washington, D. C.).

Foschi, Vittorugo. Una proprietà dei centri di curvatura delle traiettorie dei punti di una figura piana che si muove nel suo piano. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 29–31 (1943).
Au moyen de la formule de Savary l'auteur démontre le théorème [assez connu d'ailleurs]: en cinématique plane la tangente commune aux roulettes est le lieu des points pour lesquels le centre de courbure coincide avec le centre instantané de rotation. *O. Bottema* (Delft).

Foschi, Vittorugo. Una proprietà del moto di una figura piana. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 32–34 (1943).
Démonstration d'un théorème, qui n'est pas autre chose qu'une propriété connue du mouvement cardioidale [cf., par exemple, Cardinaal, Leerboek der Kinematika, Delft, 1914, p. 21]. *O. Bottema* (Delft).

Foschi, Vittorugo. La costruzione del cerchio dei flessi sul moto piano. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 35–39 (1943).
Sur quelques théorèmes très connus de la cinématique plane. L'auteur ne semble pas connaître la notion du cercle des rebroussements [cf., par exemple, Koenigs, Leçons de Cinématique, Paris, 1897, p. 153]. *O. Bottema* (Delft).

Foschi, Vittorugo. Un teorema generale nel moto di un corpo rigido intorno a un asse fisso. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 40–43 (1943).
Considérations élémentaires sur le mouvement d'un solide autour d'un axe fixe. *O. Bottema* (Delft).

Foschi, Vittorugo. Sulla distribuzione e sulle figure delle velocità e delle accelerazioni nel moto rigido piano. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 477–484 (1943).
Sur le centre instantané de rotations et sur le centre d'accélération. L'article ne contient rien de nouveaux. Les cercles considérés sont ceux de Bresse. *O. Bottema.*

Vâlcovici, Victor. Sur la distribution des axes principaux d'inertie. Mathematica, Timişoara **20**, 113–136 (1944).
Démonstration des théorèmes sur la géométrie des masses, pour la plupart très connus [cf., par exemple, Handbuch der Physik, t. 5, pp. 254–265]. L'auteur étudie en particulier la rotation des axes principaux d'inertie lorsque le centre change de position. Nous citons le théorème: les moments principaux d'inertie ne changent pas au voisinage du centre de gravité. *O. Bottema* (Delft).

Cisotti, Umberto. Punto materiale orientato. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 469–472 (1943).
Pour étudier le mouvement d'un solide sphérique homogène S, on écrit, outre l'équation du mouvement du centre de gravité, celle du moment cinétique par rapport au centre. On appelle point matériel orienté un point pourvu d'une masse et d'un coefficient (moment polaire) auquel on fera jouer le rôle du moment d'inertie par rapport à un diamètre de la sphère. On pose des équations de mouvement analogues à celles de S et on en tire quelques conséquences immédiates, par exemple pour l'énergie cinétique.
M. Brelot (Grenoble).

Nadile, Antonio. Sopra una forma semplice delle equazioni del moto dei solidi. Boll. Accad. Gioenia Sci. Nat. Catania (3) **14**, 26–32 (1940).

Meyer zur Capellen, Walther. Das physisch-elastische Doppelpendel. Z. Instrumentenkunde **61**, 213–224 (1941).
The paper is concerned with the behavior of a double pendulum composed of a distributed mass suspended in a gravitational field by a support with nonvanishing mass which elastically resists lateral motion. The rotational and quasitranslational fundamental modes are discussed for various types and positions of the connection between spring and mass. Forced vibrations are considered. The motions of two masses respectively coupled (noncollinearly) by torsional springs to a foundation and to the first mass are also treated. *G. F. Carrier* (Providence, R. I.).

Echegaray, Andrés A. Motion of a deformable system: On a problem of Poincaré. Actas Acad. Ci. Lima **9**, 169–176 (1946). (Spanish)
The equations of motion of a particle constrained to move on a given closed curve on a rotating disc are derived from a vector form of the dynamical equations due to García [same vol., 119–136 (1946); these Rev. **8**, 102].
W. Kaplan (Ann Arbor, Mich.).

Cattaneo, Carlo. Una teoria generale del moto impulsivo in presenza di vincoli unilaterali. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **14**, 619–647 (1943).
The author considers the problem of the sudden introduction of unilateral constraints in a general dynamical system. The theory is developed geometrically in the configuration space of the dynamical system, the argument being confined to smooth constraints, for which the impulse vector is perpendicular to the surfaces of constraint. There is a close analogy between the case of a general dynamical system and the impact of a particle against a smooth surface. Some applications are discussed. The author does not appear to be acquainted with the general theory of shocks developed by Z. Horák [J. École Polytech. (2), no. **28**, 15–64 (1931)]. *J. L. Synge* (Pittsburgh, Pa.).

Arrighi, Gino. Sull'equilibrio dei sistemi anolonomi. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **5**, 1–8 (1943).
Obvious necessary and sufficient conditions for equilibrium in a nonholonomic system are stated in geometric terms. The characteristic exponents for such a point of equilibrium are also investigated. *D. C. Lewis.*

Lampariello, Giovanni. Generalizzazione del teorema di Jacobi della meccanica analitica ai sistemi anolonomi. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 734–740 (1942).
A statement of the generalized Hamilton-Jacobi theorem so as to apply to nonholonomic systems. The essential advance beyond recent work of Dobronravov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **22**, 477–480 (1939)] is that it is no longer assumed that the constraints are independent of the time. *D. C. Lewis* (College Park, Md.).

Lampariello, Giovanni. Su certe identità differenziali cui soddisfano le funzioni γ delle equazioni dinamiche di Volterra-Hamel. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 12–19 (1943).

A development of the necessary tools for the proof of the generalized Hamilton-Jacobi theorem. In particular, a derivation is given of the Volterra-Hamel equations for nonholonomic dynamical systems. *D. C. Lewis.*

Lampariello, Giovanni. Generalizzazione del metodo di Hamilton-Jacobi alla dinamica dei sistemi anolonomi. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 20–28 (1943).

The final details of the proof of the generalized Hamilton-Jacobi theorem. *D. C. Lewis* (College Park, Md.).

v. Borbély, S. Über die näherungsweise hydrodynamische Bestimmung des Geschosswiderstandes. Mat. Fiz. Lapok **49**, 254–273 (1942). (Hungarian. German summary)

Hydrodynamics, Aerodynamics, Acoustics

Ballabh, Ram. Fluid motions of the type $\xi_1=\lambda_2 u_2$ etc. and $\xi_2=\lambda_1 u_1$ etc. Proc. Nat. Acad. Sci. India. Sect. A. **13**, 151–158 (1943).

A previous paper by the author [Proc. Benares Math. Soc. (N.S.) **2**, 69–79 (1940); these Rev. **3**, 283] considers two motions of a real incompressible fluid in which the vortex lines of one coincide with the streamlines of the other. This paper is a further discussion of such motions. If (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) are the vorticity vectors of the two motions, and (u_1, v_1, w_1), (u_2, v_2, w_2) the respective velocity vectors, then the conditions on the two flows are $(\xi_1, \eta_1, \zeta_1)=\lambda_2(u_2, v_2, w_2)$ and $(\xi_2, \eta_2, \zeta_2)=\lambda_1(u_1, v_1, w_1)$, where λ_1, λ_2 are functions of x, y, z, t. Several general remarks are made about such flows and an extension of Bernoulli's theorem is obtained for the motion which is the superposition of the two flows when λ_1, λ_2 are constants. The existence of flows of the type considered is shown by the example of uniplanar flows for which λ_1, λ_2 are constants. *D. Gilbarg.*

Sona, Luigi. Sulla stabilità delle configurazioni rigide di tre vortici. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **100**, 67–73 (1941).

Somogyi, A., jun. Untersuchungen über die Gestalt wenig gekrümmter Flüssigkeits-Oberflächen. Mat. Fiz. Lapok **50**, 320–369 (1943). (Hungarian. German summary)

Es wird darauf hingewiesen, dass man die Kapillaritäts-differentialgleichung an den beinahe horizontalen Teilen der Flüssigkeitsoberflächen in die Form $z_{xx}+z_{yy}=2a^{-2}z$ schreiben kann. Es werden Lösungen dieser Differentialgleichung gesucht, die gewissen gegebenen Randbedingungen entsprechen. Die Lösung wird bei zylindersymmetrischen Flüssigkeits-oberflächen durch exponentiale, bzw. durch hyperbolische Funktionen, bei rotationssymmetrischen Oberflächen durch die Bessel'schen Funktionen 0-ter Ordnung, erster und zweiter Gattung hergestellt. Es werden ferner gewisse Lösungen hergestellt, die die beinahe horizontalen Teile der Flüssigkeitsoberflächen, die in Gefässen von konvexem vieleckigem Querschnitte auftreten, zu beschreiben scheinen. Experimente haben die entsprechenden Formeln genügend bestätigt. *From the author's summary.*

Magyar, F. Geschwindigkeitsverteilungen in allgemeiner Darstellung. Z. Physik **122**, 640–647 (1944).

The author studies flow through channels. Assume that the initial distribution is the laminar, parabolic profile: $U(y)=\psi(y)=\frac{3}{2}\bar{U}\{1-(y/y_0)^2\}$. The velocity distribution which will develop in turbulent channel flow, that is, downstream, is then written as $U(y)=\psi(y)-A\phi(y)$, where $\phi(y)$ has to be chosen in such a way that it does not contribute to the mass flow through the channel. This condition plus the necessary symmetry character imposed on $\phi(y)$ suffice to determine velocity profiles as functions of one parameter $U_{max}/\bar{U}$. *H. W. Liepmann* (Pasadena, Calif.).

Southwell, R. V., and Vaisey, Gillian. Relaxation methods applied to engineering problems. XII. Fluid motions characterized by 'free' stream-lines. Philos. Trans. Roy. Soc. London. Ser. A. **240**, 117–161 (2 plates) (1946).

Haskind, M. D. Translation of bodies under the free surface of a heavy fluid of finite depth. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 67–78 (1945). (Russian. English summary) [MF 13505]

In continuation of his previous paper [same journal **8**, 287–300 (1944); these Rev. **6**, 192] the author studies the two-dimensional wave motion generated by uniform rectilinear motion of a solid body in a heavy fluid of finite depth under a free surface. The question reduces to the determination of a function $w(z)$ (the complex potential), $z=x+iy$, which satisfies the following conditions. (1) For $0>y>-h_0$, the derivative is bounded and approaches zero for $x\to+\infty$. (2) On the boundary C of the solid we have $\partial\Re w/\partial n=c\cos(n, x)$, n being the interior normal. (3) On the free boundary $\Re(dw/dz+i\gamma w)=0$ for $y=0$. (4) For $y=-h_0$, $\Im(dw/dz)=0$. Employing a method of Kočin and using the Cauchy formula expressing the velocity inside the domain by its value on the boundary, the author gives an approximate formula for $w(z)$. General formulae for the wave drag, lift force and moment of hydrodynamic forces exerted on the body are derived. Examples of approximate solutions for circular and elliptical contours with a given circulation are computed. *S. Bergman* (Cambridge, Mass.).

Haskind, M. D. Wave resistance of a solid in motion through a fluid of finite depth. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 257–264 (1945). (Russian. English summary) [MF 14037]

This paper deals with the steady three-dimensional flow of a heavy incompressible fluid past an obstacle bounded by a surface S. The fluid is also bounded by a free boundary $z=0$ and by a rigid wall $z=-h$. In the frame-work of the "linearized theory" the mathematical problem consists in finding a harmonic function φ defined within the domain occupied by the fluid and satisfying the following boundary conditions: on S, $\partial\varphi/\partial n=c\cos(n, x)$; for $z=0$, $\varphi_{xx}+\nu\varphi_z=0$; for $z=-h$, $\varphi_z=0$; grad $\varphi\to0$ for $x\to\infty$. Here c is the speed of the body and $\nu=c/g^2$, g being the acceleration of gravity. The analogous two-dimensional problem was treated by the author in the paper reviewed above. Both papers are based on the work of Kočin, who discussed the case of infinite depth ($h=\infty$) [Transactions of the conference on the theory of wave resistance, Moscow, 1937].

The author shows that the hydrodynamical forces acting on the body can be expressed in terms of a function H of two variables which was introduced by Kočin. This function depends only on the values of φ and its gradient on the surface S. The precise definition of Kočin's function and the

formulas for the forces are too complicated to be reproduced here. For some special cases the author derives approximate expressions for Kočin's function. *L. Bers.*

Haskind, M. D. Waves arising from oscillation of bodies in shallow water. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 475–480 (1946). (Russian. English summary)

The paper presents a general analysis of the wave movement arising through the oscillation of a body under the surface of a heavy fluid of finite depth. A number of formulae for wave motion are obtained: the amplitude of the waves formed, energy expended in their formation, average magnitude of the forces acting on the body for a single cycle of the oscillation. A number of examples of the approximate solution of the problem are examined. The most important is the determination of the damping coefficients for pitching and tossing of a ship.

From the author's summary.

Eckert, Ernst. Die Berechnung des Wärmeübergangs in der laminaren Grenzschicht umströmter Körper. V D I-Forschungsh. **416**, 24 pp. (3 plates) (1942).

A general procedure is developed for calculating the temperature boundary layer and the corresponding heat transfer for flow around two-dimensional bodies with laminar incompressible boundary layers. This is based on the exact solutions for the velocity and temperature boundary layers developed along the surface of a wedge, which are worked out in considerable detail. To extend the wedge solution to the more general case an equivalent wedge flow is set up at each point along the boundary by requiring the boundary layer thickness and the velocity gradient on the general surface to equal those on the equivalent wedge. This requirement gives rise to a differential equation for the boundary layer thickness involving the potential velocity field along the surface. This equation is to be solved in general by numerical or graphical methods. The momentum thickness is obtained for the velocity boundary layer and the displacement thickness for the temperature boundary layer. The procedure is illustrated and checked against other theoretical results and experimental data for flow about circular and elliptical cylinders, Joukowski airfoils and turbine blades. In each case the boundary layer thickness and heat transfer coefficient are given along the profile.

N. A. Hall (East Hartford, Conn.).

Lees, Lester, and Lin, Chia Chiao. Investigation of the stability of the laminar boundary layer in a compressible fluid. Tech. Notes Nat. Adv. Comm. Aeronaut. no. **1115**, 83 pp. (3 plates) (1946).

The paper deals with the stability of a two-dimensional flow of a perfect gas. The analysis is thus a generalization of previous work by C. C. Lin [Quart. Appl. Math. **3**, 117–142, 218–234 (1945); 277–301 (1946); these Rev. **7**, 225, 226, 346] to this more complicated case where variations of temperature, density, viscosity and heat conductivity have to be taken into account. The emphasis is on boundary layer flow at high velocities. The functions $u, v, p, \rho, T, \mu_1, \mu_2, k$ denoting velocity components, pressure, density, temperature, first and second viscosity coefficient and heat conductivity are written in the form $f(x, y, t) = f_0(y) + f_1(y)e^{i\alpha(x-ct)}$. The equations of motion are then linearized with respect to the perturbation amplitudes. Thus four linear equations remain which together with a relation from the equation of state suffice to determine the five amplitude functions of the perturbation in velocity, pressure, density and temperature. The system of equations reduces, of course, to the Orr-Sommerfeld equation for incompressible flow. The authors study first the general analytical character of the solution and the behavior of the "inviscid" and "viscid" solution. It is found that the "viscid" solution can be taken over from the incompressible case in good approximation.

The second part of the paper deals in great detail with the inviscid solutions and leads to the formulation of a criterion analogous to the Rayleigh-Tollmien theorem for incompressible flow at infinitely large Reynolds number. It is found that the gradient of the product of density and vorticity is the decisive function. If this gradient vanishes there will in general be self-excited disturbances in the inviscid flow. The third part deals with the stability with the effect of viscosity included. Methods for the computation of the stability limit of a given flow problem are presented but the actual numerical computations are left to a subsequent paper. *H. W. Liepmann* (Pasadena, Calif.).

Leray, Jean. Extension de la théorie de Prandtl à une aile de grand allongement, mais de forme quelconque. C. R. Acad. Sci. Paris **223**, 603–605 (1946).

The author undertakes the problem of the lifting line which lies in a plane but is otherwise of arbitrary shape. He states that the trailing vortices lie in the plane and are parallel to the undisturbed stream velocity. The integral relationship by which the induced velocities are calculated is taken from a thesis by S. Legras, which was not available to the reviewer. It is not clear from the present account how the author intends to avoid the difficulty of infinite induced velocities, which was discussed recently by Gail, Theilheimer, and Lin [see J. Aeronaut. Sci. **10**, 320–321 (1943); **11**, 195–196 (1944)]. *W. R. Sears.*

Panichkin, I. A. Determination on the circulation along a span of a wing in an open and semi-open flow of rectangular cross-section. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 529–536 (1946). (Russian. English summary)

Jacob, Caius. Remarques sur la méthode approchée de Tchapliguine. C. R. Acad. Sci. Paris **223**, 714–716 (1946).

This is essentially a brief abstract of the author's works [same C. R. **205**, 1365–1367 (1937); Bull. Sci. École Polytech. Timişoara **7**, 47–59, 224–244 (1937); Portugaliae Math. **1**, 209–257 (1939); Mathematica, Cluj **17**, 1–18 (1941); these Rev. **1**, 185; **3**, 220] on approximate solutions of two-dimensional perfect compressible flow in the hodograph plane. The author states that, if state a represents the state of the gas at rest, state b represents the state of the gas at some particular velocity, say the undisturbed parallel flow in the case of flow over a solid body or the velocity of the jet boundary in the case of a free jet, then there are essentially three possibilities of approximating the true isentropic pressure-volume relation: (1) replacing the curve by the tangent at a, (2) replacing the curve by the tangent at b, (3) replacing the curve by the chord joining a and b. If Γ is the exact value of the coefficient of contraction of jets issuing from a wedge-shaped region with wedge angle less than 2π and the Γ's with subscript denote the values calculated by using either one of these approximating methods, then $\Gamma_1 \leqq \Gamma_3 \leqq \Gamma \leqq \Gamma_2$. The approximation of (2), the Kármán-Tsien method, is bad for Mach number

greater than $\frac{1}{2}$. The same methods of approximation were used by the author to extend the works of Levi-Civita and H. Villat using the potential and the stream function as independent variables [same C. R. **222**, 1329–1331 (1946); these Rev. **7**, 495]. *H. S. Tsien* (Cambridge, Mass.).

Germain, Paul. Fluides compressibles. Étude directe du cas simplifié de Chaplygin. C. R. Acad. Sci. Paris **223**, 532–534 (1946).

If one assumes an imaginary fluid for which (in two-dimensional flow) $c^2=c_0^2+q^2$ (c local velocity of sound, q stream velocity, φ velocity potential) then it is known [cf. G. Braun, Ann. Physik (5) **15**, 645–676 (1932)] that $c_0\varphi$ satisfies the differential equation defining a minimal surface. Starting from Enneper's representation of a minimal surface, the author derives a flow corresponding to an arbitrary analytic function of a complex variable. On the basis of a given airfoil section, formulas are derived for the flow around a related section. The shape of the related section cannot, however, be specified in advance. The author discusses briefly the use of these fictitious flows as approximations to real subsonic compressible flows.

D. P. Ling (Murray Hill, N. J.).

Nikolsky, A. A., and Taganov, G. I. Gas motion in a local supersonic region and conditions of break of potential flow. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 481–502 (1946). (Russian. English summary)

Considérons un obstacle solide S placé dans un courant fluide plan. On sait qu'à partir d'une valeur critique du nombre M de Mach du courant il se forme une zone supersonique localisée Z. Deux régimes sont alors possibles. (I) La frontière F de Z se compose d'un arc S' du profil S et d'un arc de courbe λ le long duquel le vecteur vitesse du fluide $\mathbf{V}=\rho e^{i\omega}$ a son module ρ égal à la vitesse du son; (II) F comprend, outre S' et λ, un arc μ, le long duquel la condensation subit une discontinuité; alors, l'écoulement est dépourvu de potentiel.

Dans la première partie de leur travail, les auteurs utilisent la notion de courbe caractéristique pour démontrer quelques propriétés de ρ et de α dans Z, prolongeant ainsi les recherches de S. Christianovitch. Voici les principales conclusions, que nous énonçons, pour simplifier, sous la forme où elles s'appliquent au cas (I), bien qu'elles demeurent partiellement valables pour les régimes du type (II). Le long de λ, α est une fonction monotone de l'arc (si, du moins, α est assez régulier sur λ). Le long des caractéristiques, α et ρ sont des fonctions monotones. Lorsque S est convexe vers le courant, les variations de ρ le long de S' satisfont à certaines inégalités: en particulier, ρ est monotone le long d'une portion rectiligne de S'.

Dans la deuxième partie, les auteurs appliquent ces résultats à l'étude du problème essentiel: trouver les conditions suffisantes pour assurer l'existence du régime (II). En premier lieu, on montre que pour un nombre M et un profil convexe S donné, on peut toujours déformer S de manière à faire apparaître une singularité μ. En second lieu, les auteurs, en discutant une hypothèse de von Kármán, prouvent qu'aucune singularité μ ne peut apparaître à l'intérieur de Z du fait de l'existence à l'intérieur de Z des points doués d'accélérations infinies, si, du moins, la courbure de S' est partout bornée.

Enfin, on indique une condition qui constituerait un critère du régime (II). Le mémoire s'achève par quelques remarques sur les avantages de l'étude du phénomène dans le plan même du mouvement. *J. Kravtchenko.*

Hayes, Wallace D. Linearized supersonic flows with axial symmetry. Quart. Appl. Math. **4**, 255–261 (1946).

The linearized equation for irrotational supersonic flows with axial symmetry is considered in a system of conical coordinates, in which $t=(r/z)(M^2-1)^{\frac{1}{2}}$ is used to replace the cylindrical coordinate r; the z-axis is the axis of symmetry and M is the Mach number of the basic uniform flow. By separation of variables, solutions are found from which more general flows can be obtained by superposition. The basic solutions are expressed in terms of hypergeometric functions and Legendre functions about $t^2=1$ and are classified into six types according to the exponents which arise in the solutions. Generating equations are given by means of which new solutions can be obtained. Solutions are shown which correspond to certain flows, such as conical flows at small angle of attack and simple lifting systems.

D. Gilbarg (Bloomington, Ind.).

Tsien, Hsue-shen. Similarity laws of hypersonic flows. J. Math. Phys. Mass. Inst. Tech. **25**, 247–251 (1946).

The author starts from the equations of two-dimensional irrotational flow. Assuming the free stream velocity V to be large not only with respect to the perturbation velocities but also with respect to free stream sound velocity ($M\gg 1$), he derives an equation for the potential φ in which terms up to the second order are retained. Nondimensional coordinates and a corresponding potential f are introduced (in terms of the chord length and the thickness ratio τ). The assumption that τ is small leads to the result that the differential equation for f depends on the single parameter $K=M\tau$, while the boundary conditions depend only on K and the thickness distribution. Thus if two thin (symmetrical) profiles having the same chord and proportional thicknesses are put in hypersonic flows such that the product $M\tau$ is the same for both, the two flows are governed by the same function f.

The author then shows that, for thin profiles having the same thickness distribution, C_DM^2 is a function of K alone in a hypersonic flow, while for angles of attack proportional to τ, C_LM^2 depends only on K. A similar result holds for axially symmetric bodies: for two bodies having proportional cross-sectional areas at corresponding points along their length, C_DM^2 is a function of K only. *D. P. Ling.*

Tsien, Hsue-Shen, and Kuo, Yung-Huai. Two-dimensional irrotational mixed subsonic and supersonic flow of a compressible fluid and the upper critical Mach number. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 995, 129 pp. (16 plates) (1946).

In the case of two-dimensional steady irrotational motion of a compressible inviscid fluid under no body force, if $u+iv=qe^{i\theta}$ is the velocity, the motion can be solved in the (u, v) or hodograph plane by a stream function $\Psi(q, \theta)$ or by an associated potential $\chi(q, \theta)=xu+yv-\varphi$, where φ is the velocity potential. The differential equations satisfied by Ψ, χ are linear, so their general solution can be built up by superposition of particular integrals. Adopting the types $\Psi(q, \theta)=q^\nu\Psi_\nu(q)e^{i\nu\theta}$, $\chi(q, \theta)=q^\nu\chi_\nu(q)e^{i\nu\theta}$, it is found that Ψ_ν, χ_ν satisfy the hypergeometric equation, which has, of course, two linearly independent hypergeometric functions as solutions. The authors insist that previous investigators have used only one type of hypergeometric functions. It is

by their use of both types that they are able to investigate the field of mixed sub- and supersonic flow, and to indicate a method of obtaining the upper critical Mach number at which "limiting lines" occur and the irrotationality of the flow breaks down.

The difficulty of slow convergence, inherent in Chaplygin's original method, is here overcome by using the asymptotic properties of the hypergeometric functions. The result is the separation of the solution in the hodograph plane into two parts. One part is of closed form and is in fact the product of a universal function of the velocity and the same solution as for incompressible flow, but with a velocity distortion or velocity correction. The other part is an infinite series which converges rapidly everywhere except in a small region which can be reduced to innocuous proportions. Thus the Chaplygin procedure is improved to a point where actual numerical calculations can be made without difficulty.

Eleven numerical tables of suitable hypergeometric functions are included, and numerical examples (worked by a slightly different method) based on the incompressible flow past an elliptic cylinder of thickness ratio 0.6 are worked out for free stream Mach numbers of 0.6, 0.7. The numerical calculations show that at a free stream Mach number of 0.6, irrotational supersonic flow exists up to a local Mach number of 1.25, whereas breakdown occurs at 1.22 for a Mach number of 0.7. *L. M. Milne-Thomson.*

Frankl, F. J. Influence of the acceleration of elongated bodies of revolution upon the resistance of the gas. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 521–524 (1946). (Russian. English summary)

Increase of resistance caused by acceleration of a body in an incompressible fluid may be found by means of the inertia coefficient. The present paper deals with the influence in a compressible gas of the speed of sound on this additional resistance. Employing the method of retarded potential, the relative increase of wave resistance caused by acceleration is found to be of the order bl/v^2 (where b is acceleration, l is the length of the "head" of the projectile, v is the velocity), if the velocity of the projectile is of the same order as the speed of sound. *Author's summary.*

Riabouchinsky, Dimitri. Remarques sur la théorie des mouvements subsoniques, soniques et supersoniques d'un fluide compressible. C. R. Acad. Sci. Paris **223**, 842–844 (1946).

Wang, Chi-Teh. On the velocity distribution of turbulent flow in pipes and channels of constant cross section. J. Appl. Mech. **13**, A-85–A-90 (1946).

The author is concerned with an improvement in the semi-empirical determination of the turbulent velocity distribution for constant area pipes. The same technique used by Prandtl [Z. Verein. Deutsch. Ingenieure **77**, 105–113 (1933)] and von Kármán [Nachr. Ges. Wiss. Göttingen **1930**, 58–76] in obtaining velocity distribution from assumed mixing length characteristics is followed. The radial mixing length variation is fitted, however, by an empirical curve satisfying boundary conditions both at the wall and at the middle of the stream. There accordingly results a velocity distribution more closely fitting the experimental data of Nikuradse [V. D. I. Forschungsh. **361**, 1–22 (1933)]. Since the wall boundary condition satisfied by the mixing length is identical with that of Prandtl and von Kármán, the same skin friction formula is obtained. *N. A. Hall.*

Batchelor, G. K. The theory of axisymmetric turbulence. Proc. Roy. Soc. London. Ser. A. **186**, 480–502 (1946).

The author gives a comprehensive development of the theory of homogeneous axisymmetric turbulence in the manner of von Kármán and Howarth for isotropic turbulence. In practice, the axis of symmetry is usually the direction of mean flow. The analysis is developed with the aid of invariant theory, as suggested in a previous paper by Robertson. The form of the fundamental velocity correlation tensor is obtained and studied in some detail. Its description requires four correlation functions connected by two differential relations. These functions depend both on the distance between the two points where correlation is considered, and on the angle which the vector joining these points makes with the axis of symmetry.

Scales of axisymmetrical turbulence are defined. Equations are obtained for the propagation of the fundamental velocity correlations and for the time rates of change of the mean squares of the velocity and the vorticity components. It is found that the rates of change of the intensity of velocity fluctuations involve two terms, the first representing viscous dissipation, the second representing a transfer of energy from one component to the other due to the finite correlation between the velocity and pressure at neighbouring points. It is concluded that the effect of the velocity-pressure correlation is to bring the two velocity components towards equality. The effect of the viscous dissipation will only be towards equality if an inequality between the curvatures at the origin of two particular velocity correlation coefficient curves, both of which are measurable, is obeyed. The tendency toward equality of the velocity components is suggested as a probable indication of tendency toward isotropy. *C. C. Lin* (Providence, R. I.).

Dean, W. R. Note on waves on the surface of running water. Proc. Cambridge Philos. Soc. **43**, 96–99 (1947).

The usual treatment of waves on the surface of running water [H. Lamb, Hydrodynamics, 6th ed., Cambridge University Press, 1932, §§ 242–244] employs the artifice of a special coefficient μ of viscosity and the final solution is obtained by making $\mu \to 0$. Since the indeterminacy of the solution with $\mu = 0$ is caused by the free surface wave and the solution can be fixed by the condition of zero surface elevation far ahead of the disturbance, the author solves the problem as such with the aid of functions of a complex variable. *H. S. Tsien* (Cambridge, Mass.).

Bondi, H. Waves on the surface of a compressible liquid. Proc. Cambridge Philos. Soc. **43**, 75–95 (1947).

Let the velocity of propagation of infinitesimal disturbances in a nonviscous fluid medium of infinite extent be c. The author's problem is to find whether there are waves with velocity greater than c if the fluid region (x, y, z) is semi-infinite with one plane boundary $z=0$, where the boundary condition (with t the time) is

$$L(\partial/\partial t, \partial^2/\partial x^2 + \partial^2/\partial y^2, \partial/\partial z)\phi = 0, \qquad z = 0;$$

ϕ is the potential satisfying the equation of propagation,

$$\text{(A)} \qquad c^{-2}\partial^2\phi/\partial t^2 = \nabla^2\phi - \alpha^2\phi,$$

where α is a constant. The author shows that the necessary conditions for faster-than-sound propagations are (I) L contains both even and odd powers of $\partial/\partial z$, (II) the equations (A) and $L\phi = 0$ have common real characteristics. A general outline of the mathematical procedure for solving the problem is followed by the detailed solutions of two particular cases: propagation of the gravity waves and the capillary

waves produced by a concentrated initial disturbance at $x=y=0$, $z=h$. For the gravity waves, only (I) is satisfied, hence there is no faster-than-sound propagation. For capillary waves, both (I) and (II) are satisfied and there is faster-than-sound propagation.

The author shows that for gravity waves the classical Cauchy-Poisson wave which was derived for $h=0$ is also valid when $h\neq 0$, especially for small time intervals after the initial disturbance is created. For longer time intervals there is the modification of "cut-off" at $\{x^2+y^2+(z+h)^2\}^{\frac{1}{2}}=ct$. For capillary waves, the viscosity of any real fluid is shown to limit the capillary wave velocity so that the faster-than-sound propagation actually does not appear. *H. S. Tsien.*

Schoch, Arnold. Betrachtungen über das Schallfeld einer Kolbenmembran. Akustische Z. 6, 318–326 (1941).

A vibrating diaphragm is set in a plane wall with the normal along the z-axis. The arguments assume that the boundary curve Γ is convex and sufficiently smooth. The analysis is familiar in classical optics and sound work. Thus, apart from the time term, the Rayleigh integral for the velocity potential Φ can be immediately transformed [this is known] to give, in the cylinder generated by Γ,

$$\Phi=C\left\{-k^{-1}\exp(kz)+(2\pi k)^{-1}\int_0^{2\pi}\exp[kr(\phi)]d\phi\right\}.$$

The first term represents a plane wave, the second the perturbation due to the boundary. Here $-ki$ is the wave number, C the diaphragm velocity and r the distance between the observation point and a point of Γ. The first term in the asymptotic expansion, for small k, of the perturbation integral is obtained in terms of the Airy integral by expanding $r(\phi)$ about the constant phase point ϕ_0, where $dr/d\phi=0$ at ϕ_0, to third order terms. The customary physical interpretation of such integrals in terms of interference suggests immediately that there are exceptional regions where the constructive interference is large, for instance, the neighborhood of the central axis for a circle as boundary curve. *D. G. Bourgin* (Urbana, Ill.).

Pekeris, C. L. Theory of propagation of sound in a half-space of variable sound velocity under conditions of formation of a shadow zone. J. Acoust. Soc. Amer. **18**, 295–315 (1946).

This paper treats the acoustic field of a point source situated in a half-space in which the sound velocity c decreases with depth. In part I a constant velocity gradient is assumed, $c=az$, z being measured upwards from the plane of zero velocity; free surface at $z=\alpha$, point source at $z=\beta$ $(0<\beta<\alpha)$. The velocity potential φ is obtained as a superposition of elementary cylindrical waves. For a unit point source of frequency $\sigma/2\pi$, in the range below the source,

$$(1)\quad \varphi=2(\beta z)^{\frac{1}{2}}e^{i\pi/4}\int_0^\infty J_0(kr)I_{in}(kz)\left\{K_{in}(k\beta)-\frac{I_{in}(k\beta)}{I_{in}(k\alpha)}K_{in}(k\alpha)\right\}kdk,$$

where $n=(\sigma^2/a^2-\frac{1}{4})^{\frac{1}{2}}$, r is horizontal distance from the point source and J, I, K are cylinder functions. [If in (1) z, β are interchanged, the potential in $\beta<z<\alpha$ is obtained, by reciprocity.] For $\alpha\to\infty$ (1) reduces to an elementary function. Multipoles can be treated in this special case also (expansions in terms of Legendre functions). Conventional methods of complex function theory are applied in order to transform (1) into a Fourier-Bessel series

$$(2)\quad \varphi=2\alpha^{-2}(\beta z)^{\frac{1}{2}}e^{i\pi/4}\sum_s\frac{K_0(rx/\alpha)J_{in}(zx/\alpha)J_{in}(\beta x/\alpha)}{[J'_{in}(x)]^2},$$

where the summation extends over the roots of $J_{in}(x)=0$ lying in the first quadrant. With a view to numerical calculation extensive use is made of asymptotic expressions for the cylinder functions involved (n being large). The series (2) is strongly convergent in the "shadow zone" in which no direct ray can penetrate (in the sense of geometrical optics). The diffraction in the shadow zone is studied in detail with respect to its dependence on r and σ; the results are interpreted physically, the attenuation in the shadow zone being described as an effect of the free surface.

In part II of the paper a general velocity distribution is assumed; the theory is a formal extension of that in part I. Cylinder functions of order $\frac{1}{3}$ are involved in the case of some simple velocity distributions. The complex zeros of $J_{in}(z)$ are investigated in an appendix.

C. J. Bouwkamp (Eindhoven).

Feshbach, Herman, and Harris, Cyril M. The effect of non-uniform wall distributions of absorbing material on the acoustics of rooms. J. Acoust. Soc. Amer. **18**, 472–487 (1946).

This paper deals with the effect of absorbing patches of simple shapes on the walls of rectangular rooms upon the modes of decay and upon the corresponding pressure distribution in the room. The absorption is given by the acoustic admittance of the material, that is, the ratio of normal velocity to excess pressure on its surface. Starting from the wave equation for the excess pressure and from the normal modes corresponding to the case without any absorption, the effect of absorption is introduced in the form of a suitable source distribution along the absorbing surface corresponding to its given local admittance. This necessitates the use of a Green's function. The resulting pressure distribution is found for each mode in the form of a linear nonhomogeneous integral equation. This equation may be solved approximately by treating the integral term as a relatively small disturbance. A triply infinite series expression is used for the Green's function, leaving out any convergence discussions, though no satisfactory results are known to the reviewer in this respect. Summing the triple series once leaves a doubly infinite series expression as a first approximation to the modal pressure distribution. This is then applied to some special problems: absorbing strips on one wall and absorbing wall long compared with the other room dimensions (duct-shaped room). Numerical results are obtained by summing the series in question with the aid of known sums. In the latter case an ordinary homogeneous second order differential equation for the modal excess pressure is derived from the integral equation by the use of a suitable delta function operating on the integral. From the pressure distribution along the absorbing wall the damping constants of the complex eigen-frequencies are calculated. The concept of effective uniform admittance is introduced as an equivalent to the nonuniform admittance along the wall. Finally, experimental tests are described, confirming the theoretical results. *M. J. O. Strutt.*

Friedlander, F. G. The diffraction of sound pulses. I. Diffraction by a semi-infinite plane. Proc. Roy. Soc. London. Ser. A. **186**, 322–344 (1946).

The author discusses the change which a sound pulse undergoes when it is diffracted by a semi-infinite screen with a straight edge. The incident pulse is moving in such a fashion that its wave front is parallel to the plane of the screen and the motion is assumed two-dimensional. Several types of pulses are considered. It is remarked that the

problem has several features in common with Sommerfeld's early diffraction theory [Math. Ann. 47, 317–374 (1895)]. *A. E. Heins* (Pittsburgh, Pa.).

Friedlander, F. G. The diffraction of sound pulses. II. Diffraction by an infinite wedge. Proc. Roy. Soc. London. Ser. A. 186, 344–351 (1946).

This is a discussion of the problem described in the preceding review, but now the geometric region is an infinite wedge whose vertex angle may be a special rational multiple of π. Sommerfeld's solution of the equivalent geometric electromagnetic problem is used here [Z. Math. Phys. 46, 11–97 (1901)]. *A. E. Heins* (Pittsburgh, Pa.).

Friedlander, F. G. The diffraction of sound pulses. III. Note on an integral occurring in the theory of diffraction by a semi-infinite screen. Proc. Roy. Soc. London. Ser. A. 186, 352–355 (1946).

[Cf. the two preceding reviews.] The author gives approximations to integrals which arose in the first paper of this series. *A. E. Heins* (Pittsburgh, Pa.).

Friedlander, F. G. The diffraction of sound pulses. IV. On a paradox in the theory of reflexion. Proc. Roy. Soc. London. Ser. A. 186, 356–367 (1946).

[Cf. the three preceding reviews.] Sommerfeld's diffraction theory is used to explain the paradox of the doubling of pressure at an infinite reflecting plane parallel to which a pulse is travelling, if this is now thought of as a limiting case of reflection. Since an infinite plane is a mathematical approximation, reflection theory applies only to those parts of the plane far removed from its edges and fails when the angle between the direction of propagation of the incident pulse and the plane becomes small. In this case, the diffracted pulse which arises from the presence of the edge of the screen must be taken into account. *A. E. Heins.*

Elasticity, Plasticity

Schützer, Walter. On a molecular theory of elastic forces. Anais Acad. Brasil. Ci. 18, 103–112 (1946). (Portuguese)

The author points out that the relations of Cauchy between the 21 constants of the theory of elastic deformations of crystals are with few exceptions contradicted by experiments. He shows that a theory of crystals based on the ideas of M. Schönberg does not imply the relations of Cauchy. The crystal is considered in this theory as consisting of a regular arrangement of mass points P. The forces acting on the P's are assumed to be central forces F; besides this a moment M is assumed to act on each P. Consequently the P's are subject to translational and angular accelerations. The existence of a generalized potential V, depending on the position and on the angular velocities, is assumed and an expression of F and M in terms of V is derived. An application of Hooke's law shows that the Cauchy relations do not subsist, in general, in this theory. *I. Opatowski* (Ann Arbor, Mich.).

Signorini, A. Trasformazioni termoelastiche finite. I. Ann. Mat. Pura Appl. (4) 22, 33–143 (1943).

This is the first part of a systematic exposition of the theory of finite deformations of an elastic solid. It consists of three chapters, the first of which is devoted to the characterization of finite deformations (including finite deformations which depend on a parameter). Chapter II is devoted to the mechanics and thermodynamics of finite deformations and chapter III discusses systems to which the concept of free energy is applicable and introduces the concept of elastically isotropic media. Hooke's law connecting the stress and strain tensors is given for isotropic media both in the Lagrangian and Eulerian forms. *F. D. Murnaghan* (Baltimore, Md.).

Signorini, A. Sul calcolo delle reazioni vincolari per un solido pesante appoggiato a un suolo orizzontale. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 2, 292–320 (1941).

Matteuzzi, Alfonso. Sui teoremi di reciprocità nei fenomeni non stazionari. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 101, 69–87 (1942).

We establish, with the use of the Laplace transform, some theorems of reciprocity for the dynamics of elastic bodies and for electromagnetism. *From the author's introduction.*

Pastori, M. Distorsioni elastiche. Rend. Sem. Mat. Fis. Milano 14, 171–191 (1940).

Expository article.

Tolotti, Carlo. Le equazioni lagrangiane della meccanica dei sistemi continui in coordinate generali. Rend. Accad. Sci. Fis. Mat. Napoli (4) 13, 69–77 (1945).

Oldroyd, J. G. A rational formulation of the equations of plastic flow for a Bingham solid. Proc. Cambridge Philos. Soc. 43, 100–105 (1947).

Denoting the deviations of stress, strain and velocity strain by p'_{ik}, e'_{ik} and e'_{ik}, respectively, and introducing the intensity of stress $p=(\frac{1}{2}p'_{ik}p'_{ik})^{\frac{1}{2}}$, the author proposes the following general stress-strain relations for a Bingham solid: $p'_{ik}=2\mu e'_{ik}$ for $p\leqq\theta$, $p'_{ik}=2\eta e'_{ik}$ for $p\geqq\theta$. Here θ denotes the yield value of the material, μ the modulus of rigidity and $\eta=\eta_1(1-\theta/p)^{-1}$, where η_1 is a constant. It is stated that "plastic flow of a Bingham solid occurs in such a manner that the rate of dissipation of energy is a minimum," but the boundary value problem for which this variational principle is supposed to furnish the solution is not clearly defined. *W. Prager* (Providence, R. I.).

Gross, B. On relaxation phenomena. Anais Acad. Brasil. Ci. 18, 129–131 (1946).

The paper is concerned with the connection between the following characteristic functions of a viscoelastic material: impedance function, dynamic modulus function, creep and relaxation functions, and distribution functions of relaxation times and of retardation times of elastic compliance. It is shown that the method of the iterated Laplace transform leads to simple formulas which allow one to determine any one of these functions when any other one is known. *W. Prager* (Providence, R. I.).

Ilyushin, A. A. Theory of small elastico-plastic deformations. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 347–356 (1946). (Russian. English summary)

The paper is a continuation of the author's earlier study [same journal 9, 207–218 (1945); these Rev. 7, 144] of the relations between the theories of plastic flow [Saint Venant, von Mises, Lévy, etc.] and the theories of plastic deforma-

tion [Hencky, Nadai]. In the earlier paper the author pointed out that the theories of these two groups furnish identical predictions whenever the so-called tensor of similarity is independent of time throughout the plastic body under consideration. In the present paper he claims that this condition is fulfilled if all exterior loads (body forces and surface stresses) increase linearly with time. This statement is actually proved, however, only in the special case where the secant modulus, and hence the tangent modulus, are proportional to powers of the octahedral shearing stress (restricted theorem). The author remarks that the actual curve of secant modulus versus octahedral shearing stress can be segmentwise approximated by powers, and argues that the aforementioned theorem should therefore be approximately valid for a wide class of plastic materials (general theorem). This argument does not seem convincing, however. *W. Prager* (Providence, R. I.).

Galin, L. A. Plane elastico-plastic problem. Plastic zones in the vicinity of circular apertures. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 367–386 (1946). (Russian. English summary)

The paper is concerned with the stresses in an infinite elastic-plastic slab with a circular hole when the state of stress at infinity and the (uniform) pressure acting on the boundary of the hole are given. The line separating the plastic from the elastic region is determined approximately by function-theoretical methods under the tacit assumption that this line includes the hole. [This assumption is not necessarily justified.] *W. Prager* (Providence, R. I.).

Galin, L. A. Spatial contact problems of the theory of elasticity for punches of circular shape in plane. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 425–448 (1946). (Russian. English summary)

The paper is devoted to an analysis of the distribution of stresses along the surface of contact between an elastic half-space and a rigid body (termed punch) when the latter is pressed against the half-space with a specified force. One of the objects of the investigation is to establish the relation between the magnitude of the force and the displacement of the punch. The punch is assumed to be a circular cylinder whose contact surface is of arbitrary shape. The case of frictionless contact is reduced to a solution of the problem of Neumann, which is solved by constructing a suitable Green's function. The case when friction is present is also considered. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Sokolovsky, W. W. Equations of the plane plastic stressed state according to the Mises theory and their approximate representation. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 357–366 (1946). (Russian. English summary)

In recent papers [C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 175–178, 421–424 (1946); these Rev. **8**, 114] the author discussed states of plane stress in plastic bodies which obey the yield conditions of von Mises or Saint Venant. If the nonvanishing principal stresses are denoted by σ_1 and σ_2, the yield condition of von Mises is represented by an ellipse in the (σ_1, σ_2)-plane, the yield condition of Saint Venant by an inscribed hexagon. In the present paper, an intermediate yield condition is used which is represented by a dodecagon inscribed in the Mises ellipse. This yield condition is stated to be sufficiently close to the experimentally verified condition of von Mises, but more convenient for computational purposes. *W. Prager* (Providence, R. I.).

Sokolovsky, W. W. On a problem of elastico-plastic bending of plates. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 13–16 (1946).

The paper is concerned with the elastic-plastic stresses in a simply supported circular plate under a uniformly distributed load. The plate material is supposed to be perfectly plastic and to obey Hencky's stress-strain relations. The spreading of the plastic regions with increasing load and the dependence of the central deflection on the intensity of the load are studied. *W. Prager* (Providence, R. I.).

Donnell, L. H., Drucker, D. C., Goodier, J. N., and Reissner, Eric. The effect of transverse shear deformation on the bending of elastic plates. J. Appl. Mech. **13**, A-249–A-252 (1946).

This is a discussion of a paper by E. Reissner with the same title [same J. **12**, A-69–A-77 (1945); these Rev. **7**, 42]. The first and second authors examine problems similar in type to that treated by Reissner but from a different point of view, while the third gives experimental results. *J. J. Stoker* (New York, N. Y.).

Federhofer, K. Die dünne Kreisringplatte mit grosser Ausbiegung. Österreich. Ing.-Arch. **1**, 21–35 (1946).

The object of this investigation is to determine whether the membrane theory can be applied to a study of deflections of thin plates in the form of a circular ring loaded along its inner edge and hinged along its outer edge. It is demonstrated that the theory, which does not take into account the stiffness of the plate, gives an incorrect picture of deflection except for the cases where the ratio of the radius of the hole to the radius of the plate exceeds 0.7. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Sen, Bibhutibhusan. Two-dimensional boundary-value problems of elasticity. Proc. Roy. Soc. London. Ser. A. **187**, 87–101 (1946).

In this paper [written in 1941] there are solved in closed form, with the help of complex functions, the following problems of plane stress. (1) Isotropic half plane with (a) a single force acting in the interior, (b) a concentrated twisting couple in the interior, (c) a point of pressure in the interior. (2) Orthotropic half plane with (a) twisting couple, (b) point of pressure. (3) Isotropic circular region with (a) two self-equilibrating concentrated forces in the interior, (b) two twisting couples in the interior. It is stated that (1a, b) have previously been solved by E. Melan [Z. Angew. Math. Mech. **12**, 343–346 (1932)] and S. Ghosh [Bull. Calcutta Math. Soc. **29**, 177–184 (1937)], respectively, and that the remaining solutions are believed to be new. The method of the paper, which appears to be somewhat different from other complex variable methods used in the theory of plane stress, has been given before by the author [Philos. Mag. (7) **26**, 98–119 (1938); **27**, 596–604 (1939)]. *E. Reissner* (Cambridge, Mass.).

Sen, Bibhutibhusan. Boundary value problems of a heavy circular disk held in a vertical plane. Philos. Mag. (7) **37**, 66–72 (1946).

Goldenweiser, A. L. Procedures of integration of equations of the theory of thin shells. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 387–396 (1946). (Russian. English summary)

The author gives a method of determining the edge effect along those contours of the shell whose tangents do not

coincide with the asymptotic lines of the middle surface. The integration of the system of equations of the shell theory is reduced, in several cases, to the integration of momentless theory. This paper is a sequel to the author's earlier paper [same journal **9**, 463–478 (1945); these Rev. **7**, 351]. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Rabotnov, J. N. Bending of a cylindrical shell under a concentrated load. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 299–300 (1946).

The author applies his equations for thin shells [same C. R. **47**, 89–90, 329–331 (1945); these Rev. **7**, 142] to the bending of a cylindrical shell under a concentrated load. However, the investigations of E. Reissner [J. Math. Phys. Mass. Inst. Tech. **23**, 184–191 (1944); these Rev. **6**, 195] show that, for points near the concentrated load, the effects of transverse shear cannot be neglected. The author neglects these effects; thus the calculation seems to be unsatisfactory. *H. S. Tsien* (Cambridge, Mass.).

Green, A. E. The flexure and torsion of aeolotropic beams. Proc. Cambridge Philos. Soc. **43**, 68–74 (1947).

A general Saint Venant theory is given for a type of aeolotropic beam of constant cross section under the combined action of flexure and torsion. The beams possess three mutually perpendicular planes of elastic symmetry and nine coefficients of elasticity are present. Following usual methods the problem is reduced to the determination of a stress function which in turn is decomposed into a combination of seven flexure or torsion functions each harmonic in a cross section obtained by a linear deformation of the section of the beam. However, the mapping generally is not a conformal one, and modified boundary conditions are imposed on the normal derivatives of these functions. It is stated that these flexure functions can be found in some cases by the methods of R. M. Morris [Proc. London Math. Soc. (2) **46**, 81–98 (1940); **49**, 1–18 (1945); these Rev. **1**, 189; **7**, 501]. Results are given for the case of pure torsion of an elliptic aeolotropic prism. *D. L. Holl.*

Klinger, Friedrich. Der elastische Schwerpunkt am räumlichen Bogenträger. II. Akad. Wiss. Wien, S.-B. IIa. **149**, 447–469 (1940).

Extensions of results of part I [same vol., 269–289 (1940); these Rev. **3**, 288].

Girkmann, Karl. Gleichgewichtsverzweigung an einem querbelasteten Druckstabe. Akad. Wiss. Wien, S.-B. IIa. **150**, 257–279 (1941).

Funk, P. Stabilitätstheorie bei Stäben unter Druck und Drillung. Österreich. Ing.-Arch. **1**, 2–14 (1946).

The author investigates the stability of thin straight cylindrical rods under the action of axial pressures and twisting moments applied at the ends. The theory is developed by variational methods under rather general assumptions. For example, it is not assumed at the outset that the bending and twisting moments in the rod are the same in deflected positions of the rod as in the straight position. The stability criteria are obtained as the conditions under which the second variation of the potential energy of the rod ceases to be positive definite. This in turn leads to an eigenvalue problem for a system of homogeneous ordinary differential equations (the Jacobi equations) from which the critical values of the loads are obtained in the usual manner. The author treats in detail the special cases in which the bending stiffness is the same for all axes through the centroid of the cross section area of the rod. The Jacobi equations for the variations $X(s)$, $Y(s)$ normal to the axis of the rod can be combined in this case into the single differential equation $Ad^4Z/ds^4-iMd^3Z/ds^3+Pd^2Z/ds^2=0$ for the complex quantity $Z=X+iY$. The quantities A, M and P are the bending stiffness, twisting moment, and thrust, respectively. The boundary conditions (which are always homogeneous, of course) can also be expressed in terms of Z. The author proceeds to give the stability criteria in a number of cases which had been treated previously [P. Fillunger, Z. Angew. Math. Mech. **6**, 294–308 (1926); M. Born, Göttingen dissertation, 1906; E. L. Nicolai, Z. Angew. Math. Mech. **6**, 30–43 (1926)]. He then treats the following new case. The rod is considered to be vertical and to have horizontal rods or "handles" attached rigidly at its ends. The rod is then twisted by applying appropriate forces to the handles. The critical twisting moment is determined under the assumption that the handles remain in a horizontal plane, while the originally straight vertical rod may turn about the handles when it becomes unstable and buckles. The critical twisting moment will then depend on the initial angle between the handles at the two ends. The author finds that the critical twisting moment has its minimum value when the initial angle between the handles is either 15.12° or 105.12°. *J. J. Stoker.*

Galli, Adriano. Complementi analitici utili per la trattazione effettiva dei problemi di stabilità dell'equilibrio elastico. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 152–170 (1942).

Pignedoli, Antonio. Sulle vibrazioni di una piastra circolare sollecitata al contorno da una pressione radiale pulsante. Atti Soc. Nat. Mat. Modena (6) **75**, 180–197 (1944).

Federhofer, Karl. Über den Einfluss von Ungenauigkeiten der Form und Stärke eines Kreisringes auf die Schwingzahlen seiner ebenen Biegungsschwingungen. Akad. Wiss. Wien, S.-B. IIa. **150**, 117–130 (1941).

The natural vibrational frequency of a complete elastic thin circular ring is modified by the presence of variations in the ring cross-section and in the curvature of the central fiber. Under the approximation that the eigenfunction is sinusoidal (having just one harmonic) in the angular coordinate, the eigen-frequency is found with the aid of Hamilton's principle. Numerical results are tabulated for the case where the nonuniformities are also sinusoidal. *G. F. Carrier.*

Udeschini, Paolo. Onde di discontinuità nei corpi elastoplastici. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **14**, 651–659 (1943).

The paper is concerned with the motion of a surface of discontinuity in an elastic medium which obeys a slightly nonlinear stress-strain law. [Following the practice accepted in the recent Italian literature, the author calls such a material "mezzo lievemente elastoplastico." While justified in certain special cases, this is apt to cause confusion.] As the deviation from the linear stress-strain law makes itself felt, the longitudinal waves cease to be purely longitudinal, while the transverse waves split up into two groups; the first of these are purely transverse, the second only approximately so. *W. Prager* (Providence, R. I.).

Mindlin, J. A. Propagation of waves over the surface of a circular cylinder of infinite length. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 107–110 (1946).

The author considers axially symmetric waves moving over the surface of a circular cylinder of infinite length and, in particular, deduces that all such waves with wave length less than a certain upper bound propagate at velocities lying between the transversal wave velocity and the Rayleigh wave velocity. The results indicate that waves also exist whose propagation velocity for a given axial wave length depends on the radial "wave length" and hence are not strictly surface waves. *G. F. Carrier.*

García Beltrán, Miguel. On the parametric representation of internal elastic waves in anisotropic media of the rhombic crystalline system. Revista Ci., Lima **48**, 33–63 (1 plate) (1946). (Spanish)

The author applies the parametric methods of Rudzki to elastic wave propagation in orthorhombic and hexagonal crystals. Choosing the axes of symmetry as coordinate axes he obtains the parametric equation of the wave front in tangential coordinates. The curve corresponding to the parametric equation in each coordinate plane is found to possess three concentric branches. The first branch gives the parametric equations of an ellipse, so that the wave surface is an ellipsoid. In crystals belonging to the orthorhombic system, it is an ellipsoid with three unequal axes. In hexagonal crystals it is an ellipsoid of revolution and in the cubic system it is a sphere. The second branch is represented by an equation of the sixth degree with 24 cuspidal points, of which 8 are real, and 28 double points, of which 4 are real. The corresponding wave front is a new kind of Riemann surface of class 4, order 12, genus 3 according to Plücker's classification. It is of the general type studied by F. Klein. The third branch of the meridional curve is a transform of the second and represents a deformed oval, with no real singular points. The author applies his theory to topaz, barite and Rochelle salt. The paper is somewhat marred by typographical errors, which, however, do not obscure the argument, and by a strange and somewhat misleading choice of historical references in the introduction. The most commonly observed seismic surface waves are those of Love and Rayleigh; not Lamb, Wiechert, Uller, or Nakano waves as the author seems to state.

J. B. Macelwane (St. Louis, Mo.).

Altschuler, L. V. Sur l'explosion dans un milieu compressible plastique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 199–202 (1946).

The propagation of a spherical explosion wave in a plastic body is discussed under the following simplifying assumptions: (1) the density varies but slightly, (2) the bulk modulus is constant, (3) the material flows under constant maximum shearing stress. *W. Prager.*

Rakhmatulin, K. A. Propagation of unloading waves along a bar with variable elastic limits (accumulation of residual deformation). Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 333–346 (1946). (Russian. English summary)

The paper is concerned with the following problem: the end of a semi-infinite cylindrical bar of an elastic-plastic material is subjected to repeated longitudinal blows; to study the accumulation of permanent deformation along the bar. *W. Prager* (Providence, R. I.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Herzberger, Max. The limitations of optical image formation. Ann. New York Acad. Sci. **48**, 1–29 (1946).

The author restates his new diapoint method of handling skew rays [J. Opt. Soc. Amer. **26**, 197–204 (1936)], and applies it to discuss some fundamental questions in the geometrical optics of rotationally symmetrical systems. The method employs, together with the concept of diapoints, an iconal function called the diapoint characteristic, obtained by introducing new variables into Hamilton's angle-characteristic.

A pair of diapoints P, P' is the pair of intersections, in the object and image space, respectively, of an arbitrary skew ray traced through the system with any meridional plane through the axis of the system. The doubly infinite set of skew rays starting from a given object-point P and traversing the system will meet the meridional plane through P in a doubly infinite set of points P', the set of diapoints of P. From the properties of the set of points P', all lying in a single meridional plane, the properties of the image of P by the system can be deduced; if the points P' all coincide, then P is sharply imaged by the system; if they all lie along a small segment of a straight line, then P is said to be "symmetrically imaged"; if along a short arc of a curve, then P is said to be "half-symmetrically imaged."

In the case where the object is at infinity, the coordinates of the diapoints P' are expressible in terms of the first partial derivatives of an iconal function $N(\zeta, \beta, \zeta')$ with respect to its three arguments ζ, β, ζ', just as in Hamilton's theory the coordinates of the intersection of the skew ray with the image plane are expressible in terms of the derivatives of the angle characteristic T. The condition for a coma-free system is now that $\partial N/\partial \zeta'$ shall vanish for the axial object-point at infinity; the conditions for a half-symmetrical image and for a sharp image can also be expressed very simply, in terms of the second-order partial derivatives of N. In the case of a finite object-point, the diapoints of given diamagnification m (the diamagnification is defined as the ratio of the distances of the diapoints P and P' from the axis of the system) are investigated by means of a new function, the diapoint characteristic $D(u, v, w)$; D, like N, is obtained from T by introducing new variables, but in the case of D these new variables depend on m.

The theory is applied to discuss two general problems, (a) how many surfaces can be sharply imaged by an optical system, (b) in what circumstances can a system image every point of the object space half-symmetrically or symmetrically?

In the last two sections of the paper, the diapoint characteristic $D(u, v, w)$ is evaluated explicitly for a single refracting surface, not necessarily spherical, and a brief indication is given of the way in which the theory can be applied to a general discussion of optical aberrations. As the author points out, some of his results have previously been obtained, by other methods, jointly with H. Boegehold [Compositio Math. **1**, 448–476 (1935); Z. Angew. Math. Mech. **15**, 157–178 (1935)] and by Boegehold [Z. Instrumentenkunde **56**, 98–109 (1936)]. *E. H. Linfoot.*

König, H. Die Ähnlichkeitsgesetze des elektromagnetischen Feldes und ihre Anwendung auf Hohlraumresonatoren. Hochfrequenztech. Elektroak. **58**, 174–180 (1941).

This paper consists of an investigation of geometrically similar systems of conductors and dielectrics. More precisely the author determines the conditions under which a transformation of the type $(x', y', z') = m^{-1}(x, y, z)$; $E'(x', y', z') = aE(x, y, z)$; $H'(x', y', z') = bH(x, y, z)$; $\epsilon'(x', y', z') = \alpha\epsilon(x, y, z)$; $\mu'(x', y', z') = \beta\mu(x, y, z)$; $\sigma'(x', y', z') = \gamma\sigma(x, y, z)$; and $t' = dt$ leaves the Maxwell equations unchanged. If S and S' are two similar resonators for which the linear dimensions of the latter are smaller by a factor of $1/m$ than those of S, if the conductivity of S' is greater by a factor of m than the conductivity of S, and if μ and ϵ are the same in both resonators, then the eigenfrequencies of S' are related to those of S by $\omega' = m\omega$ and the field in S' at time $t' = m^{-1}t$ is similar to that in S at the time t. The damping coefficients for the two systems are equal.

However, if, as is usually the case, the conductivities of the two resonators are equal and large, then the fields inside the metal are no longer similar although the fields in the dielectric are essentially similar. The damping coefficient in S' is now larger than that of S by a factor of $m^{\frac{1}{2}}$.

R. S. Phillips (New York, N. Y.).

Magnus, W., und Oberhettinger, F. Die Berechnung des Wellenwiderstandes einer Bandleitung mit kreisförmigem bzw. rechteckigem Aussenleiterquerschnitt. Arch. Electrotechnik **37**, 380–390 (1943). [MF 16300]

The author develops a method for determining the principal-wave equivalent line impedance for an infinite uniform coaxial conductor. The actual computation is carried out for a circular cross-section outer conductor with an inner conductor consisting of a linear band lying on a diameter, and for a rectangular outer conductor with a linear band inner conductor. The problem is equivalent to finding an analytic function $w = u(z) + i\,v(z)$ with real part $u(z)$ constant on each of the conductor surfaces. By assuming that one can determine from symmetry considerations the curves $v(z) = v(z_1)$ and $v(z) = v(z_2)$, where z_1 and z_2 are end points of the inner conductor band, the problem is reduced to that of finding a conformal mapping of a simply connected region into the indicated rectangle. For the two examples stated, the latter mapping is accomplished by means of elliptic functions. *R. S. Phillips* (New York, N. Y.).

Fischer, J. Stromverdrängung im zylindrischen Leiter, insbesondere von elliptischem Querschnitt. Phys. Z. **42**, 327–336 (1941).

The author formulates the following set of postulates for a transverse-magnetic quasi-stationary field of an infinitely long conductor of uniform cross-section. (1) $H_z = 0$, $\partial/\partial z = 0$, the vector potential has only the component $A_z = \Re(Ae^{i\omega t})$. Here z is in the axial direction. (2) Outside the conductor $\Delta A = 0$; inside the conductor $\Delta A = k^2 A$, where $k^2 = i\omega\sigma\mu$ and σ is the conductivity and μ the permeability of the conductor. (3) On the surface of the conductor A and its first derivatives are continuous. (4) For large distances r from the conductor, A decreases as $\ln r$. Much of the previous work on this problem replaced the condition (3) by the convenient but inaccurate assumption of a current constant along the conductor surface. A conductor with elliptical cross-section is treated in detail. The solution is expressed in terms of a series of Mathieu functions. Various methods are suggested for approximating the coefficients of this series.

R. S. Phillips (New York, N. Y.).

Terletzky, J. Induction of fast charged particles currents by rotating magnetized cosmic bodies. Acad. Sci. USSR. J. Phys. **10**, 377–382 (1946).

The motions of electric charges in the electromagnetic field of a rotating magnetic dipole are considered. It is pointed out that such particles accumulate energy in the course of their motion, unless the direction of the magnetic dipole moment is parallel to the axis of rotation. Expressions are found for the energy gained by particles in the magnetic fields of the earth and of the sun. It is shown that such particles spiral round the lines of magnetic force and are "blown off" by the electric component. Unless starting from a region near the pole, they do not leave the body altogether but return together with the line of force. Particles starting near the poles, however, may leave the body and acquire energies of the same order as that of cosmic ray particles. The author is thus in agreement with Alfvén's theories of the origin of cosmic rays.

It is pointed out, however, that particles accelerated by magnetised cosmic bodies may account not only for cosmic rays, but also, in the fields of the sun or earth, for a number of geophysical and heliophysical phenomena. In particular, attention is drawn to the photons emitted by these particles.

L. Jánossy (Manchester).

Marié, Pierre. Sur le filtrage des ondes. C. R. Acad. Sci. Paris **223**, 352–354 (1946).

Marié, Pierre. Propagation des ondes dans les systèmes périodiques, compte tenu de certaines conditions aux limites. C. R. Acad. Sci. **222**, 1039–1042 (1946).

In the first of these two notes the author applies the method he developed [same C. R. **222**, 869–870 (1946); these Rev. **8**, 186], using electrospherical polynomials for studying electrical wave filters with a central portion of iterated identical four-terminal reactive nets, to the propagation of electromagnetic waves through a series of lossless thin parallel plane obstacles, which except for the first and last are identical and equally spaced, and which have given coefficients of reflection and transmission. Such a system is shown to behave like a filter, with frequency pass-bands depending on the spacing. A formula is given for the attenuation ratio.

In the second note the method is shown to apply to a series of parallel plane interfaces separating regions of alternating refractive index in the optical case, or alternating characteristic impedance in the case of wave guides. Here except for the first and last the interfaces are spaced the same number of wavelengths apart, the wavelength alternating in alternate sections. Again the system acts like a filter. There is a possible application to the propagation of X-rays through crystals. *O. Frink* (State College, Pa.).

Duffin, R. J. Nonlinear networks. I. Bull. Amer. Math. Soc. **52**, 833–838 (1946).

If in a linear network containing resistors, inductors and capacitors there is no undamped free vibration then, aside from transient terms which die out, the system has a unique response to a given impressed voltage. The author extends this result to the case where the resistors need not be linear. His result in the case of a single mesh circuit described by $L\ddot{q} + V(\dot{q}) + Sq = e(t)$ requires that $\delta \leqq V'(\dot{q}) \leqq \Delta$, where δ and Δ are positive constants. Actually the results are stated for a general network in terms of matrix functions. [It follows from known results that a much weaker condition on $V(\dot{q})$ suffices for uniqueness in the one mesh case, suggesting the possibility of a less restrictive result in the general case.]

N. Levinson (Cambridge, Mass.).